Solving Interval Linear Systems Is NP-Hard Even When All Inputs Are Known With the Same Accuracy*

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Abstract

It is known that in general, solving interval linear systems is NP-hard. There exist several proofs of this NP-hardness, and all these proofs use examples with intervals of different width – corresponding to different accuracy in measuring different coefficients. For some classes of interval linear systems with the same accuracy, feasible algorithms are known. We show, however, that in general, solving interval linear systems is NP-hard even when all inputs are known with the same accuracy.

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1 Introduction

Interval computations are needed. In practice, we only know approximate values of physical quantities x: e.g., we only know measurement results $\widetilde{x}$, and measurements are never absolutely accurate; see, e.g., [6]. In many practical situations, the only information that we have about the measurement accuracy is the upper bound Δ on the absolute value of the measurement error $\Delta x \stackrel{\text{def}}{=} \widetilde{x} - x$: $|\Delta x| \le \Delta$. In such situations, based on the measurement result $\widetilde{x}$, we can only conclude that the (unknown) actual value x belongs to the interval $\boldsymbol{x} = [\underline{x}, \overline{x}] \stackrel{\text{def}}{=} [\widetilde{x} - \Delta, \widetilde{x} + \Delta]$.

If we have such interval information $\boldsymbol{x}_i = [\underline{x}_i, \overline{x}_i]$ about the quantities x_i ($i = 1, \ldots, n$), and we want to estimate the value of a quantity y which is related to x_i by a known relation $y = f(x_1, \ldots, x_n)$, then we can only conclude that y belongs to the range $\boldsymbol{y} = \{f(x_1, \ldots, x_n) : x_1 \in \boldsymbol{x}_1, \ldots, x_n \in \boldsymbol{x}_n\}$. Computing this range is one of the main problems of *interval computations* (see, e.g., [4]).

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Interval computations are, in general, NP-hard. In general, the problem of computing the endpoints $\underline{y}$ and $\overline{y}$ of the desired range $\boldsymbol{y} = [\underline{y}, \overline{y}]$ is NP-hard, even when the function $f(x_1, \ldots, x_n)$ is polynomial (even when the function $f(x_1, \ldots, x_n)$ is quadratic); see, e.g., [1, 3].

In particular, the problem of interval computations is NP-hard in a practically useful case when the algorithm $f(x_1, \ldots, x_n)$ corresponds to solving a system of linear equations. In this case:

- we know that the quantities $y_1, \ldots, y_m$ satisfy a system of linear equations $\sum\limits_{j=1}^{m} a_{ij} \cdot y_j = b_i$, $i = 1, 2, \ldots$,
- we know intervals of possible values $\boldsymbol{a}_{ij}$ and $\boldsymbol{b}_i$ for a_{ij} and b_j, and
- we want to find the interval of all possible values of, e.g., y_1 corresponding to different combinations of values of $a_{ij} \in \boldsymbol{a}_{ij}$ and $b_i \in \boldsymbol{b}_i$.

Discussion. The values of some of the coefficients a_{ij} and b_i come from measurements and are, thus, only known with interval uncertainty. In some cases, however, we know that b_i is not affected by y_j and thus, the corresponding coefficient a_{ij} is equal to 0; in such cases, no measurement is needed.

For example, when we try to reconstruct the values of the actual signal $x(t)$ at different moments of time based on the sensor recordings $\widetilde{x}(t)$ at the same moments of time, then we know, from causality, that the value $\widetilde{x}(t)$ cannot be affected by the future values $x(s)$ with $s > t$. In this case, $a_{ts} = 0$.

It is worth mentioning that this situation is different from the case when in principle, the dependence is possible, but the corresponding coefficient a_{ij} is so small that our measuring instruments are unable to detect this dependence. In this case, the measured value $\widetilde{a}_{ij}$ is 0, but, due to the measurement uncertainty, the actual value a_{ij} may be non-zero.

What if all the measurements have the same accuracy. The original proof that interval computations is NP-hard used situations in which all intervals $\boldsymbol{x}_i$ have the same width, i.e., in measurement terms, when all measurements were performed with the same accuracy; see, e.g., [1, 3].

However, in the known proofs that solving interval linear systems is NP-hard, different intervals $\boldsymbol{a}_{ij}$ and $\boldsymbol{b}_i$ have different widths. In some cases when all the widths are equal, it is possible to find a feasible algorithm for solving the corresponding interval linear systems (even explicit formulas); see, e.g., [2]. It is therefore reasonable to ask whether the problem of solving interval linear systems remains NP-hard when all inputs are known with the same accuracy.

What we do. In this paper, we prove that the problem of solving interval linear systems is NP-hard even if we restrict ourselves to systems in which all the inputs are known with the same accuracy Δ. To be more precise, we prove that for every accuracy $\Delta > 0$, the problem is NP-hard even if we only consider systems in which all the input intervals have the same half-width Δ.

2 Results

Definition 1.

- *By an* interval linear system*, we mean a tuple consisting of integers m and n and intervals $\boldsymbol{a}_{ij}$ and $\boldsymbol{b}_i$, $1 \le i \le n$, $1 \le j \le m$ with rational bounds. A system will also be denoted by* $\sum\limits_{j=1}^{m} \boldsymbol{a}_{ij} \cdot y_j = \boldsymbol{b}_i$.
- *We say that a tuple $y = (y_1, \ldots, y_m)$ is a* possible solution *to the interval linear system if for some $a_{ij} \in \boldsymbol{a}_{ij}$ and $b_i \in \boldsymbol{b}_i$, we have* $\sum\limits_{j=1}^{m} a_{ij} \cdot y_j = b_i$ *for all i.*
- *By the* problem of solving interval linear systems*, we mean the following problem:*
  - *given an interval linear system and a rational number $\varepsilon > 0$,*
  - *find ε-approximations to the endpoints $\underline{y}_1$ and $\overline{y}_1$ of the range of all the values y_1 corresponding to all possible solutions $(y_1, \ldots, y_m)$ of the given interval linear system.*

Definition 2. *Let $\Delta > 0$ be a rational number. We say that an interval linear system* $\sum\limits_{j=1}^{m} \boldsymbol{a}_{ij} \cdot y_j = \boldsymbol{b}_i$ *is* uniformly Δ-accurate *if each interval $\boldsymbol{a}_{ij}$ or $\boldsymbol{b}_i$ is either identically 0 or has half-width Δ.*

Proposition 1. *For every $\Delta > 0$, the problem of solving uniformly Δ-accurate interval linear systems is NP-hard.*

Comment. It turns that out that not just computing the range of solutions is NP-hard, even checking whether a given system has solutions at all is NP-hard.

Proposition 2. *For every $\Delta > 0$, it is NP-hard to check whether a uniformly Δ-accurate interval linear system has a possible solution.*

Comment. Proposition 2 is similar to Theorem 22.5 from [3], according to which it is NP-hard to check, given a matrix $\widetilde{a}_{ij}$, whether all matrices a_{ij} with

$$a_{ij} \in [\widetilde{a}_{ij} - 1, \widetilde{a}_{ij} + 1]$$

are regular. This similarly can be enhanced, since from Theorem 22.5, we can extract the following corollary: for every rational $\Delta > 0$, it is NP-hard to check, given a matrix $\widetilde{a}_{ij}$, where all matrices a_{ij} with $a_{ij} \in [\widetilde{a}_{ij} - \Delta, \widetilde{a}_{ij} + \Delta]$ are regular.

3 Proof of Propositions 1 and 2

1°. By definition (see, e.g., [3, 5]), a problem P_0 is NP-hard is every problem from the class NP can be reduced to this problem P_0. Thus, to prove that a given problem P_g is NP-hard, it is sufficient to prove that a known NP-hard problem P_k can be reduced to P_g: in this way, every problem from the class NP can be reduced to P_k, and P_k can be reduced to P_g, so every problem from the class NP can be reduced to P_g.

As such a problem P_k, we take the following *subset sum* problem (see, e.g., [3, 5]):

- given positive integers $s_1, \ldots, s_n$,
- find the values $\varepsilon_i \in \{-1, 1\}$ for which $\sum\limits_{i=1}^{n} \varepsilon_i \cdot s_i = 0$.

2°. To prove Proposition 1, we will reduce each instance $(s_1, \ldots, s_n)$ of the subset sum problem to the following interval linear system with $m = n + 1$ unknowns $y_1, \ldots, y_n, y_{n+1}$:

- for each i from 1 to n, we add two interval linear equations

$$[\Delta, 3\Delta] \cdot y_i + [-\Delta, \Delta] \cdot y_{n+1} = 0; \quad (1)$$

$$[-\Delta, \Delta] \cdot y_i + [-3\Delta, -\Delta] \cdot y_{n+1} = 0; \quad (2)$$

- we also add two additional equations

$$[1, 1 + 2\Delta] \cdot y_{n+1} = [-\Delta, \Delta]; \quad (3)$$

$$\sum_{i=1}^{n} [M \cdot s_i - \Delta, M \cdot s_i + \Delta] \cdot y_i = 0, \quad (4)$$

where we denoted $M \stackrel{\text{def}}{=} 3\Delta \cdot n$.

One can easily check that in this system, all inputs are indeed known with the same accuracy Δ.

Let us prove the following two implications:

- if the original instance of the subset sum has a solution, then the range $\boldsymbol{y}_1$ of possible values of y_1 is equal to $[-\Delta, \Delta]$;
- on the other hand, if the original instance of the subset problem does not have a solution, then the $\boldsymbol{y}_1 = [0, 0]$.

Thus, if we can compute the endpoints of the interval $\boldsymbol{y}_1$ with accuracy $\varepsilon < \Delta/2$, we will be able to tell whether a given instance of the subset problem has a solution; thus, we will have the desired reduction.

3°. To prove the above implications, let us first analyze the system (1)–(4).

Equation (1) means that $a_i \cdot y_i = b_i \cdot y_{n+1}$ for some $a_i \in [\Delta, 3\Delta]$ and $b_i \in [-\Delta, \Delta]$. Thus, $y_i = r_i \cdot y_{n+1}$, where the coefficient $r_i \stackrel{\text{def}}{=} b_i/a_i$ takes a value from the interval $[-\Delta, \Delta]/[\Delta, 3\Delta] = [-1, 1]$. So, $|r_i| \le 1$.

Equation (2) means that $a'_i \cdot y_i = b'_i \cdot y_{n+1}$ for some $a'_i \in [-\Delta, \Delta]$ and $b'_i \in [\Delta, 3\Delta]$. Here, $|a'_i| \le \Delta$ and $|b'_i| \ge \Delta$. Substituting $y_i = r_i \cdot y_{n+1}$, with $|r| \le 1$, into this equation, we conclude that

$$(a'_i \cdot r_i) \cdot y_{n+1} = b'_i \cdot y_{n+1}. \quad (5)$$

3.1°. We either have $y_{n+1} = 0$ or $y_{n+1} \ne 0$.

If $y_{n+1} = 0$, then from $y_i = r_i \cdot y_{n+1}$, we conclude that $y_i = 0$ for all i. In this case, we have a tuple consisting of all zeros. One can check that this tuple is a possible solution of the system (1)–(4).

3.2°. If $y_{n+1} \ne 0$, then, dividing both sides of the equation (5) by y_{n+1}, we conclude that

$$a'_i \cdot r_i = b'_i. \quad (6)$$

Since $b'_i \geq \Delta$, we cannot have $a'_i = 0$. If we had $|r_i| < 1$, then we would have $|a'_i \cdot r_i| < |a'_i| \leq \Delta$, which contradicts to the fact that for $b'_i = a'_i \cdot r_i$, we have $|b'_i| \geq \Delta$. Since $|r_i| \leq 1$ and it is not possible to have $|r_i| < 1$, we conclude that $|r_i| = 1$, i.e., that $y_i = r_i \cdot y_{n+1}$ for some $r_i \in \{-1, 1\}$. Thus, all possible solutions $(y_1, \ldots, y_{n+1})$ with $y_{n+1} \neq 0$ have the form $y_i = \pm y_{n+1}$.

4°. From the equation (3), it follows that $|y_{n+1}| \leq \Delta$. Since $y_1 = r_1 \cdot y_{n+1}$ for $r_1 = \pm 1$, we conclude that $|y_1| \leq \Delta$ for all possible solutions $(y_1, \ldots, y_{n+1})$.

5°. The equation (4) means for some α_i for which $|\alpha_i| \leq \Delta$, we have

$$\sum_{i=1}^{n} (M \cdot s_i + \alpha_i) \cdot y_i = 0,$$

i.e.,

$$M \cdot \sum_{i=1}^{n} s_i \cdot y_i = -\sum_{i=1}^{n} \alpha_i \cdot y_i.$$

Substituting $y_i = r_i \cdot y_{n+1}$ into the formula (6) and dividing both sides by $y_{n+1} \neq 0$, we conclude that

$$M \cdot \sum_{i=1}^{n} r_i \cdot s_i = -\sum_{i=1}^{n} \alpha_i \cdot r_i. \tag{7}$$

Since $|\alpha_i| \leq \Delta$ and $r_i = \pm 1$, we have

$$\left| \sum_{i=1}^{n} \alpha_i \cdot r_i \right| \leq \sum_{i=1}^{n} |\alpha_i| \leq n \cdot \Delta.$$

Thus, from (7), we get

$$M \cdot \left| \sum_{i=1}^{n} r_i \cdot s_i \right| \leq n \cdot \Delta. \tag{8}$$

Dividing both sides of this inequality by $M = 3\Delta \cdot n$, we conclude that

$$\left| \sum_{i=1}^{n} r_i \cdot s_i \right| \leq \frac{1}{3}. \tag{9}$$

The values s_i are integers, the values $r_i = \pm 1$ are also integers, so the sum $\sum\limits_{i=1}^{n} r_i \cdot s_i$ is also an integer. The fact that the absolute value of this integer does not exceed 1/3 means that this integer is equal to 0, i.e., that $\sum\limits_{i=1}^{n} r_i \cdot s_i = 0$.

Thus, if the system (1)–(4) has a non-zero possible solution, then the original instance of the subset problem has a solution.

6°. From the previous statement, we can conclude that if the original instance has no solutions, then the only possible solution to the system (1)–(4) is an all-zeros solution. In this case, the range $\boldsymbol{y}_1$ is equal to $[0, 0]$.

7°. If the original instance has a solution $\varepsilon_i \in \{-1, 1\}$ for which $\sum\limits_{i=1}^{n} \varepsilon_i \cdot s_i = 0$, then, for each value $y_1 \in [-\Delta, \Delta]$, we can take $y_i = \dfrac{\varepsilon_i}{\varepsilon_1} \cdot y_1$ for all $i \leq n$ and $y_{n+1} = \dfrac{y_1}{\varepsilon_1}$. One can easily check that these values form a possible solution of the system (1)–(4); indeed:

- The equation (1) is satisfied since for these values y_i, we have

$$\Delta \cdot y_i + (-\Delta \cdot \varepsilon_i) \cdot y_{n+1} = 0,$$

with $\Delta \in [\Delta, 3\Delta]$ and $-\Delta \cdot \varepsilon_i \in [-\Delta, \Delta]$.

- The equation (2) is satisfied since we have

$$(\Delta \cdot \varepsilon_i) \cdot y_i + (-\Delta) \cdot y_{n+1} = 0,$$

with $\Delta \cdot \varepsilon_i \in [-\Delta, \Delta]$ and $-\Delta \in [-3\Delta, -\Delta]$.

- Equation (3) is satisfied since

$$1 \cdot y_{n+1} = y_{n+1},$$

where $1 \in [1, 1 + 2\Delta]$ and $y_{n+1} \in [-\Delta, \Delta]$.

- Finally, the equation (4) is satisfied since we have

$$\sum_{i=1}^{n} (M \cdot s_i) \cdot y_i = 0,$$

with $M \cdot s_i \in [M \cdot s_i - \Delta, M \cdot s_i + \Delta]$.

On the other hand, we know that for all possible solutions, we have $|y_1| \le \Delta$. Thus, in this case, the desired range $\boldsymbol{y}_1$ of possible values of y_1 coincides with the the interval $[-\Delta, \Delta]$.

The reduction is proven, and so is Proposition 1.

8°. To prove Proposition 2, we will reduce each instance $(s_1, \ldots, s_n)$ of the subset sum problem to the following interval linear system with $m = n + 1$ unknowns $y_1, \ldots, y_n, y_{n+1}$:

- for each i from 1 to n, we add two interval linear equations

$$[\Delta, 3\Delta] \cdot y_i = [-\Delta, \Delta]; \quad (10)$$

$$[-\Delta, \Delta] \cdot y_i = [\Delta, 3\Delta]; \quad (11)$$

- we also add an additional equation

$$\sum_{i=1}^{n} [M \cdot s_i - \Delta, M \cdot s_i + \Delta] \cdot y_i = [-\Delta, \Delta], \quad (12)$$

where we denoted $M \stackrel{\text{def}}{=} 3\Delta \cdot (n + 1)$.

One can easily check that in this system, all inputs are indeed known with the same accuracy Δ.

Let us prove the following two implications:

- if the original instance of the subset sum has a solution, then the system (1)–(4) has a possible solution;
- on the other hand, if the original instance of the subset problem does not have a solution, then the system (1)–(4) does not have a possible solution.

9°. To prove the above implications, let us analyze the system (10)–(12).

9.1°. Equation (10) implies that $y_i \in [-\Delta, \Delta]/[\Delta, 3\Delta] = [-1, 1]$. So, $|y_i| \le 1$.

9.2°. Equation (11) means that $a_i \cdot y_i = b_i$ for some $a_i \in [-\Delta, \Delta]$ and $b_i \in [\Delta, 3\Delta]$. Here, $|a_i| \le \Delta$ and $b_i \ge \Delta$. Since $b_i \ge \Delta$, we cannot have $a_i = 0$.

If we had $|y_i| < 1$, then we would have $|a_i \cdot y_i| < |a_i| \le \Delta$, which contradicts to the fact that for $b_i = a_i \cdot y_i$, we have $b_i \ge \Delta$. Since $|y_i| \le 1$ and it is not possible to have $|y_i| < 1$, we conclude that $|y_i| = 1$, i.e., that $y_i \in \{-1, 1\}$.

Thus, all possible solutions $(y_1, \ldots, y_n)$ of the system (10)–(12) have the form $y_i = \pm 1$.

9.3°. The equation (12) means for some α_i and α_0 for which $|\alpha_i| \le \Delta$, we have

$$\sum_{i=1}^{n} (M \cdot s_i + \alpha_i) \cdot y_i = \alpha_0,$$

i.e.,

$$M \cdot \sum_{i=1}^{n} s_i \cdot y_i = \alpha_0 - \sum_{i=1}^{n} \alpha_i \cdot y_i. \tag{13}$$

Since $|\alpha_i| \le \Delta$ and $y_i = \pm 1$, we have

$$\left| \alpha_0 - \sum_{i=1}^{n} \alpha_i \cdot y_i \right| \le \sum_{i=0}^{n} |\alpha_i| \le (n+1) \cdot \Delta.$$

Thus, from (13), we get

$$M \cdot \left| \sum_{i=1}^{n} y_i \cdot s_i \right| \le (n+1) \cdot \Delta. \tag{14}$$

Dividing both sides of this inequality by $M = 3\Delta \cdot (n+1)$, we conclude that

$$\left| \sum_{i=1}^{n} y_i \cdot s_i \right| \le \frac{1}{3}. \tag{14}$$

The values s_i are integers, the values $y_i = \pm 1$ are also integers, so the sum $\sum\limits_{i=1}^{n} y_i \cdot s_i$ is also an integer. The fact that the absolute value of this integer does not exceed 1/3 means that this integer is equal to 0, i.e., that $\sum\limits_{i=1}^{n} y_i \cdot s_i = 0$.

10°. Thus, if the system (1)–(4) has a possible solution, then the original instance of the subset problem has a solution.

11°. To complete the proof of Proposition 2, let us show that if the original instance of the subset sum problem has a solution ε_i, then the system (10)–(12) also has a solution $y_i = \varepsilon_i$. Indeed:

- Equation (10) is satisfied for every i, since we have

  $$(\Delta \cdot \varepsilon_i) \cdot y_i = \Delta,$$

  where $\Delta \cdot \varepsilon_i \in [-\Delta, \Delta]$ and $\Delta \in [-\Delta, \Delta]$.

- Equation (11) is satisfied, since we have

$$\Delta \cdot y_i = (\Delta \cdot \varepsilon_i),$$

  with $\Delta \in [\Delta, 3\Delta]$ and $\Delta \cdot \varepsilon_i \in [-\Delta, \Delta]$.

- Finally, the equation (12) is satisfied, since we have

$$\sum_{i=1}^{n} (M \cdot s_i) \cdot y_i = 0,$$

  with $M \cdot s_i \in [M \cdot s_i - \Delta, M \cdot s_i + \Delta]$ and $0 \in [-\Delta, \Delta]$.

The reduction is proven, and so is Proposition 2.

Comment. In the above reductions, the number of equations is, in general larger than the number of unknowns; however, we can easily make these two numbers equal if we add extra unknowns that do not affect equations at all. Thus, the problem remains NP-hard even if we limit ourselves to square systems.

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