Images are Easier to Restore than 1-D Signals: A Theoretical Explanation of a Surprising Empirical Phenomenon

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Abstract

Similar techniques are often used to restore 1-D signals and 2-D images from distorted ("blurred") observations. From the purely mathematical viewpoint, 1-D signals are simpler, so it should be easier to restore signals than images. However, in practice, it is often easier to restore a 2-D image than to restore a 1-D signal. In this paper, we provide a theoretical explanation for this surprising empirical phenomenon. ©2014 World Academic Press, UK. All rights reserved.

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1 Formulation of the Problem: A Surprising Empirical Phenomenon

Empirical fact. In his presentations at the IEEE World Congress on Computational Intelligence WCCI'2012 (Brisbane, Australia, June 10–15, 2012), J. M. Mendel mentioned a puzzling fact [5]:

- replacing usual type-1 fuzzy techniques [2, 7, 10] with type-2 techniques (see, e.g., [4, 6]) often drastically improves the quality of 2-D *image* processing,

- on the other hand, similar type-2 methods, in general, do not lead to any significant improvement in processing 1-D *signals* (e.g., in predicting time series).

This is not just a weird property of type-2 techniques: J. Mendel recalled that he encountered a similar phenomenon in the 1980s and early 1990s, when he was applying more traditional statistical methods to processing geophysical signals and images; see, e.g., [3].

Why this is surprising. From the purely mathematical viewpoint, a 1-D signal means that we have intensity values depending only on one variable (time), while a 2-D image means that we have intensity values depending on two variables – namely, on two spatial coordinates. From this viewpoint, signals are a simplified 1-D version of the 2-D images. It is therefore natural to expect that it is easier to reconstruct a 1-D signal than a 2-D image – but this is not what we observe.

What we do in this paper. In this paper, we provide a theoretical explanation for the above surprisingly empirical phenomenon.

Comment. This justification is an additional argument that a picture is indeed worth a thousand words :-)

2 Main Idea Behind Our Theoretical Explanation of the Observed Phenomenon

General description of distortion. Both in signal and in image processing, the observed signal is somewhat distorted ("blurred"):

- for signals, the observed value $\widetilde{x}(t)$ at a moment t depends not only on the actual signal $x(t)$ at this moment of time, but also on the values of the signal $x(t')$ at nearby moments of time t';

- similarly, for images, the value $\widetilde{I}(x, y)$ that we observe at a 2-D point with coordinates (x, y) depends not only on the actual intensity $I(x, y)$ at this spatial point, but also on spatial intensities $I(x', y')$ at nearby points (x', y').

Both in signal processing and in image processing, this distortion is usually well-described as *convolution* (see, e.g., [8]), i.e., as a transformation from $x(t)$ to

$$\widetilde{x}(t) = \int K(t - t') \cdot x(t') \, dt' \tag{1}$$

and from $I(x, y)$ to

$$\widetilde{I}(x, y) = \int K(x - x', y - y') \cdot I(x', y') \, dx' \, dy'. \tag{2}$$

Our goal is to reconstruct the original signal $x(t)$ (or the original image $I(x, y)$) from the distorted observations $\widetilde{x}(t)$ (or $\widetilde{I}(x, y)$).

An additional complication is that the functions $K(t)$ (or $K(x, y)$) which describe the distortion are not known exactly.

What we prove. In the next section, we show that in some reasonable sense, it is easier to restore a 2-D image than to restore a 1-D signal. In precise terms, we prove that when we do not have any information about the distortion function, then, in the ideal no-noise case:

- it is, in general, *not possible* to uniquely reconstruct the original 1-D signal;

- however, it is, in general, *possible* to uniquely reconstruct the original 2-D image.

Comment. In real life, when noise *is* present, the reconstruction is, of course, no longer unique, but the above empirical fact shows that, with some accuracy, reconstruction of 2-D images is still possible.

3   Our Theoretical Explanation: Technical Details

Convolution can be naturally described in terms of Fourier transform. It is well known (see, e.g., [1, 8, 9]) that formula for the convolution can be simplified if we use Fourier transform

$$\widehat{x}(\omega) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} x(t) \cdot \exp(\mathrm{i} \cdot \omega \cdot t) \, dt, \tag{3}$$

$$\widehat{I}(\omega_x, \omega_y) = \frac{1}{2\pi} \cdot \int \int I(x, y) \cdot \exp(\mathrm{i} \cdot (\omega_x \cdot x + \omega_y \cdot y)) \, dx \, dy. \tag{4}$$

Namely, in terms of Fourier transforms, the formulas (1) and (2) take a simple form

$$\widehat{\widetilde{x}}(\omega) = \widehat{K}(\omega) \cdot \widehat{x}(\omega), \tag{5}$$

$$\widehat{\widetilde{I}}(\omega_x, \omega_y) = \widehat{K}(\omega_x, \omega_y) \cdot \widehat{I}(\omega_x, \omega_y). \tag{6}$$

In practice, we only observe discrete signals (images). In practice, we only observe finitely many intensity values. For a signal, we measure the values $\widetilde{x}_k$ corresponding to moments $t_k = t_0 + k \cdot \Delta$, where Δt is the time interval between two consecutive measurements. For an image, we similarly usually measures intensities $\widetilde{I}_{k,\ell}$ corresponding to the grid points $(x_k, y_\ell) = (x_0 + k \cdot \Delta x, y_0 + \ell \cdot \Delta y)$.

In the discrete case, Fourier transforms can be reformulated in terms of polynomials. Based on the observed discrete value, we cannot recover the original signal (image) with high spatial resolution, we can only hope to recover the values $x_k \stackrel{\text{def}}{=} x(t_k)$ and $I_{k,\ell} \stackrel{\text{def}}{=} I(x_k, y_\ell)$ the original signal (image) on the same grid (or an even sparser set. In terms of the observed and actual grid values, the Fourier transform formulas take the form of an integral sums, such as

$$\widehat{\widetilde{x}}(\omega) = \frac{1}{\sqrt{2\pi}} \cdot \sum_{k=0}^{N} (\widetilde{x}_k \cdot \Delta t) \cdot \exp(\mathrm{i} \cdot \omega \cdot k \cdot \Delta t). \tag{7}$$

In terms of $s \stackrel{\text{def}}{=} \exp(\mathrm{i} \cdot \omega \cdot \Delta t)$, this formula takes the polynomial form $\widehat{\widetilde{x}}(\omega) = P_{\widetilde{x}}(s)$, where

$$P_{\widetilde{x}}(s) \stackrel{\text{def}}{=} \sum_{k=0}^{N} (\widetilde{x}_k \cdot \Delta t) \cdot s^k. \tag{8}$$

Similarly, we have $\widehat{x}(\omega) = P_x(s)$ and $\widehat{K}(\omega) = P_K(s)$, where

$$P_x(s) \stackrel{\text{def}}{=} \sum_{k=0}^{N} (x_k \cdot \Delta t) \cdot s^k, \quad P_K(s) \stackrel{\text{def}}{=} \sum_{k=0}^{N} (K_k \cdot \Delta t) \cdot s^k. \tag{9}$$

For these polynomials, we have

$$P_{\widetilde{x}}(s) = P_K(s) \cdot P_x(s). \tag{10}$$

This equality holds for infinitely many different values $s \stackrel{\text{def}}{=} \exp(\mathrm{i} \cdot \omega \cdot \Delta x)$ corresponding to infinitely many different values ω.

It is known that the difference between two polynomials of degree N is also a polynomial of the same degree and thus, this difference can have no more than N roots. So, if the difference between two polynomials is equal to 0 for infinitely many values s, this means that this difference is identically 0, i.e., that the equality (10) holds for all possible values s.

Similarly, for the 2-D image case, for $s_x \stackrel{\text{def}}{=} \exp(\mathrm{i} \cdot \omega \cdot \Delta x)$ and $s_y \stackrel{\text{def}}{=} \exp(\mathrm{i} \cdot \omega \cdot \Delta y)$, we get

$$P_{\widetilde{I}}(s_x, s_y) = P_K(s_x, s_y) \cdot P_I(s_x, s_y), \tag{11}$$

where

$$P_{\widetilde{I}}(s_x, s_y) \stackrel{\text{def}}{=} \sum_{k=0}^{N} \sum_{\ell=0}^{N} \left(\widetilde{I}_{k,\ell} \cdot \Delta x \cdot \Delta y \right) \cdot s_x^k \cdot s_y^\ell, \quad P_K(s_x, s_y) \stackrel{\text{def}}{=} \sum_{k=0}^{N} \sum_{\ell=0}^{N} (K_{k,\ell} \cdot \Delta x \cdot \Delta y) \cdot s_x^k \cdot s_y^\ell, \tag{12}$$

$$P_I(s_x, s_y) \stackrel{\text{def}}{=} \sum_{k=0}^{N} \sum_{\ell=0}^{N} (I_{k,\ell} \cdot \Delta x \cdot \Delta y) \cdot s_x^k \cdot s_y^\ell. \tag{13}$$

In terms of the resulting polynomials, reconstructing a signal (image) means factoring a polynomial. In terms of the polynomial equalities (10) and (11), the problem of reconstructing a signal or an image takes the following form: we know the product of two polynomials, and we want to reconstruct the factors that lead to this product.

In 1-D case, there are many ways to represent a polynomial as a factor of two others. In the 1-D case, each complex-valued polynomial $P_{\widetilde{x}}(s)$ of degree N has, in general, N complex roots $s^{(1)}$, $s^{(2)}$, etc., and can, therefore, be represented as $|P(s)|^2 = \text{const} \cdot (s - s^{(1)}) \cdot (s - s^{(2)}) \cdot \ldots$. In this case, there are many factors, so there are many ways to represent it as a product of two polynomials.

In the 2-D case, polynomial factorization is almost always unique.   Interestingly, in contrast to the 1-D case, in which each polynomial can be represented as a product of polynomials of 1st order, in the 2-D case, a generic polynomial *cannot* be represented as a product of polynomials of smaller degrees.

Indeed, to describe a general polynomial of two variables $\sum_{k=0}^{n} \sum_{\ell=1}^{n} c_{kl} \cdot s_x^k \cdot s_y^\ell$ in which each of the variables has a degree $\leq n$, we need to describe all possible coefficients c_{kl}. Each of the indices k and ℓ can take $n + 1$ possible values $0, 1, \ldots, n$, so overall, we need to describe $(n + 1)^2$ coefficients.

When two polynomials multiply, the degrees add: $s^m \cdot s^{m'} = s^{m+m'}$. Thus, if we represent $P(s)$ as a product of two polynomials, one of them must have a degree $m < n$, and the other one degree $n - m$. In general:

- we need $(m + 1)^2$ coefficients to describe a polynomial of degree m and

- we need $(n - m + 1)^2$ coefficients to describe a polynomial of degree $n - m$,

- so to describe arbitrary products of such polynomials, we need $(m + 1)^2 + (n - m + 1)^2$ coefficients.

To be more precise, in such a product, we can always multiply one of the polynomials by a constant and divide another one by the same constant, without changing the product. Thus, we can always assume that, e.g., in the first polynomial, the free term c_{00} is equal to 1. As a result, we need one fewer coefficient to describe a general product: $(m + 1)^2 + (n - m + 1)^2 - 1$.

To be able to represent a generic polynomial $P(s)$ of degree n as such a product $P(s) = P_m(s) \cdot P_{n-m}(s)$, we need to make sure that the coefficients at all all $(n + 1)^2$ possible degrees $s_x^k \cdot s_y^\ell$ are the same on both sides of this equation. This requirement leads to $(n + 1)^2$ equations with $(m + 1)^2 + (n - m + 1)^2 - 1$ unknowns.

In general, a system of equations is solvable if the number of equations does not exceed the number of unknowns. Thus, we must have $(n + 1)^2 \leq (m + 1)^2 + (n - m + 1)^2 - 1$. Opening parentheses, we get

$$n^2 + 2n + 1 \leq m^2 + 2m + 1 + (n - m)^2 + 2 \cdot (n - m) + 1 - 1.$$

The constant terms in both sides cancel each other, as well as the terms $2n$ in the left-hand side and $2m + 2 \cdot (n - m) = 2n$ in the right-hand side, so we get an equivalent inequality $n^2 \leq m^2 + (n - m)^2$. Opening parentheses, we get $n^2 \leq m^2 + n^2 - 2 \cdot n \cdot m + m^2$. Cancelling n^2 in both sides, we get $0 \leq 2m^2 - 2 \cdot n \cdot m$. Dividing both sides by $2m$, we get an equivalent inequality $0 \leq m - n$, which clearly contradicts to our assumption that $m < n$.

Concluding argument.   Since a generic 2-D polynomial cannot be factorized, this means that, in general, from the product $P_{\tilde{I}}(s_x, s_y)$ of two 2-variable polynomials (11), we can uniquely determine both factors – in particular, we can uniquely determine the polynomial $P_I(s_x, s_y)$.

Based on the the observed value $\tilde{I}(x, y)$, we can determine $P_{\tilde{I}}(s_x, s_y)$, and from the polynomial $P_I(s_x, s_y)$, we can uniquely determine its coefficients $I_{k,\ell} \cdot \Delta x \cdot \Delta y$, and thus, we can determine the original intensity values $I_{k,\ell} = I(x_k, y_\ell)$. So, in the absence of noise, we can indeed (almost always) uniquely reconstruct a 2-D image but not a 1-D signal. The statement is proven.

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