Journal of Uncertain Systems
Vol.8, No.x, pp.xx-xx, 2014
Online at: www.jus.org.uk

On Early Stages of Idea Propagation, the Number of Adopters Grows as $n(t) \approx c \cdot t^a$: Theoretical Explanation of the Empirical Observation

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Received 27 July 2013; Revised 14 August 2013

Abstract

New good ideas sometimes propagate too slowly. To speed up their propagation, we need to have a quantitative understanding of how ideas propagate. An intuitive understanding of ideas propagation has led to several reasonable first-approximation mathematical models. These models provide a good description of idea propagation on the later stages, when the ideas have already been adopted by a reasonably large number of people. However, at the critically important early stages, these models are not perfect: these models predict a linear growth with time, while empirical growth data is often better described by a power law. In this paper, we provide an intuitive theoretical explanation of the observed power-law growth.

Keywords: propagation of ideas, power law growth, early stage of invention adoption

1 Formulation of the Problem: Empirical Power-Law Growth Seems to Be Inconsistent with Simple Intuitive Models of Idea Propagation

Propagation of new tools and new ideas – one of the main ways science and technologies progress. Science and technology are progressing at an enormous speed. New ideas appear all the time, new tools are being designed all the time that enable us to do things that we could not do before – and do them faster, more reliably, and more efficiently.

It is extremely important to come up with new ideas, to design new tools, but mere design is not enough: it is important to make sure that these ideas and tools do not stay with their inventors, that they are widely adopted and thus propagate.

Current first approximation model of ideas propagation. We would like to know how ideas propagate, i.e., how the number $n(t)$ of people who use the new idea grows with time t.

The main current model of idea propagation (see, e.g., [3, 6]) is as follows. For the idea to spread, people who have not yet adopted the idea must learn about it – either from the original announcement or from people who already use this idea. The probability that a new person will learn this new idea can thus be estimated as $a + b \cdot n(t)$, where a is the probability to learn this idea from the original announcement, and b is the probability to encounter one of the followers. Out of the total population of N people, $N - n(t)$ not-yet-users are exposed to this learning. Since the probability of each of them learning about the new idea is proportional to $a + b \cdot n(t)$, the total number of people who learn about the new idea is proportional to $(a + b \cdot n(t)) \cdot (N - n(t))$. Thus, we arrive at the differential equation

$$\frac{dn}{dt} = c \cdot (a + b \cdot n(t)) \cdot (N - n(t)),$$

where c is the corresponding proportionality coefficient. Thus, we get

$$\frac{dn}{dt} = (A + B \cdot n(t)) \cdot (N - n(t)), \tag{1}$$

where $A \stackrel{\text{def}}{=} c \cdot a$ and $B \stackrel{\text{def}}{=} c \cdot b$.

Many refinements of this model have been proposed (see, e.g., [6]), but the model (1) remains the main first approximation model of knowledge propagation.

Solution to the first approximation model. By moving all the terms containing n to the left-hand side and all the terms containing time t to the right-hand side, we conclude that

$$\frac{dn}{(A+B\cdot n)\cdot(N-n)} = dt. \tag{2}$$

Here, as one can easily check,

$$\frac{1}{(A+B\cdot n)\cdot(N-n)} = \frac{1}{A+B\cdot N}\cdot\left(\frac{B}{A+B\cdot n}+\frac{1}{N-n}\right). \tag{3}$$

Thus, the left-hand side of the formula (2) takes the form

$$\frac{1}{A+B\cdot N}\cdot\left(\frac{B\cdot dn}{A+B\cdot n}+\frac{dn}{N-n}\right) =$$

$$\frac{1}{A+B\cdot N}\cdot\left(\frac{d(A+B\cdot n)}{A+B\cdot n}-\frac{d(N-n)}{N-n}\right). \tag{4}$$

So, the integral of this left-hand side takes the form

$$\frac{1}{A+B\cdot N}\cdot(\ln(A+B\cdot n)-\ln(N-n)) = \frac{1}{A+B\cdot N}\cdot\ln\left(\frac{A+B\cdot n}{N-n}\right). \tag{5}$$

Hence, integrating both sides of the equation (2), we get

$$\frac{1}{A+B\cdot N}\cdot\ln\left(\frac{A+B\cdot n}{N-n}\right) = t + C, \tag{6}$$

where C is the integration coefficient. Therefore,

$$\ln\left(\frac{A+B\cdot n}{N-n}\right) = k\cdot t + c, \tag{6}$$

where $k \stackrel{\text{def}}{=} A + B\cdot N$ and $c \stackrel{\text{def}}{=} k \cdot C$. Raising e to the (equal) left- and right-hand sides of the equation (6), we get

$$\frac{A+B\cdot n}{N-n} = C'\cdot\exp(k\cdot t), \tag{7}$$

where $C' \stackrel{\text{def}}{=} \exp(c)$. Multiplying both sides of this equation by $N - n$, we get

$$A + B\cdot n = N\cdot C'\cdot\exp(k\cdot t) - n\cdot C'\cdot\exp(k\cdot t), \tag{8}$$

hence

$$n\cdot(B + C'\cdot\exp(k\cdot t)) = C'\cdot\exp(k\cdot t) - A, \tag{9}$$

and, thus,

$$n(t) = \frac{C'\cdot\exp(k\cdot t) - A}{B + C'\cdot\exp(k\cdot t)}. \tag{10}$$

If we start measuring time from the moment the idea was launched, so that $n(0) = 0$, then we conclude that $C' = A$ and thus, the formula (10) takes the form

$$n(t) = \frac{A\cdot(\exp(k\cdot t) - 1)}{B + A\cdot\exp(k\cdot t)}. \tag{11}$$

Initial propagation of a new idea: asymptotic description. Once the idea has spread, it will continue spreading; the most critical period is right after the idea's appearance, when $t \approx 0$. For such t, asymptotically, the first approximation model (11) implies that

$$n(t) \approx c \cdot t, \tag{12}$$

where $c \stackrel{\text{def}}{=} \dfrac{A \cdot k}{B + A}$ i.e., that $n(t)$ linearly grows with time t.

Empirical data seems to be inconsistent with this asymptotics. While for medium and large times t, the first approximation model (11) is in a reasonably good accordance with data, for small t, the empirical data shows a clearly non-linear behavior (see, e.g., [1, 4]), a behavior which is better described by a power function

$$n(t) \approx c \cdot t^a \tag{13}$$

for some $a \neq 1$.

What we do in this paper. In this paper, we provide a simple intuitive model which explains such power-law growth.

2 Our Explanation

Main idea behind our explanation: a qualitative description. To describe how a new idea propagates, let us consider one specific tool (or idea) aimed at solving problems from a specific class. For example, this tool may be a new (e.g., more efficient) software for solving large systems of linear equations.

Any person who sometimes solves the problem from the given class is a potential user of this tool. We start with the initial situation, in which only the author of the tool knows it and uses it. Eventually, other potential users start learning and using this tool.

When does a potential user start learning the new tool? On the one hand, there are clear benefits in learning a new tool: once a person learns the new tool, he or she can solve the problems from the corresponding class more efficiently.

- For example, efficiency may mean faster computations. In this case, the user will be able to solve large systems of linear equations faster. This will save the time needed to solve such systems, and enable the user to get the results faster.
- Alternatively, efficiency may mean that the user may be able to use fewer processors of a multi-processor computer system to solve the same problem – so, if the user pays for the computer time, he or she will be able to save some money by using this new software.

On the other hand, new tools, new ideas are not always easy to learn. One needs to invest some effort – e.g., time – into learning the new tool. A potential user will start learning the new tool only if the expected benefits exceed the investment needed to learn this tool. So, to figure out when a particular user will start learning the tool, we need to be able to estimate both potential benefits and the required investment.

The potential benefits of using a tool depend on how often it will be used.

- Some potential users solve the corresponding problems very frequently. For such users, a potential benefit may be large.
- Other users encounter the corresponding problems rarely. For such users, the potential benefit of learning the new tool may be small.

In our analysis, we need to take this difference into account.

To estimate the difficulty of learning the tool, we need to take into account that this difficulty depends on how many people have already learned it. If a tool is currently used only by a few folks, it is more difficult to learn it: if there is a question about this tool, it is not so easy to find someone who knows the answer. On the other hand, if the tool is widely used, learning this tool is much easier: when there is a question, one of the nearby colleagues who is already using this tool can answer.

For example, in a Computer Science department, it is easy for someone to learn one of the widely used languages such as C++ or Java: whatever question may arise, there are plenty of people around who know these languages already. On the other hand, a new operating system – e.g., a new version of Windows – may be simpler to use than C++, but in the beginning, it is not so easy to learn – since in the beginning, very few people have an experience of using it and therefore, it is difficult to find help if a problem arises.

Let us show how these qualitative ideas can be translated into a quantitative model.

Heavy users vs. light users. As we have mentioned, a user will start learning the new tool only if the expected benefit of its use exceeds the expenses needed to learn this tool. For each user, the expected benefit b of using the tool is proportional to the number x of the corresponding problems (per unit time) that this user encounters: $b = C \cdot x$, for some proportionality constant C (that describes the benefit of using the tool to solve a single problem). From this viewpoint, each user can be characterized by the corresponding value x.

Let $L(t)$ describe the cost of learning the tool at moment t. In this notation, at each moment of time t, a potential user – characterized by the value x – will start learning the tool if the benefit $C \cdot x$ exceeds the cost $L(t)$: $C \cdot x > L(t)$. This condition can be equivalently described as $x > x_0(t)$, where we denoted $x_0(t) \stackrel{\text{def}}{=} \dfrac{L(t)}{C}$. This ratio $x_0(t)$ serves as a threshold:

- "heavy users", i.e., users for which $x > x_0(t)$, will start adopting the tool, while
- "light users", i.e., users for which $x < x_0(t)$, will continue using previous tools.

Thus, at each moment of time, the state of propagation can be characterized by a single value – this threshold value $x_0(t)$.

Distribution of users. To describe how knowledge propagates, we need to know how many users are there with different levels of usage x. In many practical problems, the distribution is described by the power law (see, e.g., [2, 7]), in which the proportion $P(x \geq X)$ of objects x for which x exceeds a given threshold X is determined by a formula

$$P(x \geq X) = C_0 \cdot X^{-\alpha} \tag{14}$$

for some constants C_0 and α. The ubiquitous character of power laws was popularized by Benoit Mandelbrot in his fractal theory; see, e.g., [5].

How easier is to to learn a new tool when we already have a given number n of users. In the beginning, learning a new tool is not very easy, but ultimately, tools and techniques become relatively easy to learn. For example:

- calculus used to be a great 17 century achievement, accessible only to a few great minds;
- however, nowadays, many kids study elements of calculus already in high school.

The reason why, in the beginning, learning a new tool is not easy is that a person learning the new tool *can* go astray (and *goes* astray). The more advice we get, the more accurately we understand what needs to be done – i.e., crudely speaking, the more accurate is the direction in which we are going – and thus, the smaller amount of effort will be wasted. The resulting amount of effort can be viewed as proportional to the inaccuracy with which we know the direction in which to go in learning the tool.

To find this direction, we can use the advice and expertise of the existing users. If we have n users that we can consult, this means that we have n estimates for the desired direction. In general, according to statistics (see, e.g., [8]), if we have n similar independent estimates of the same quantity, then, by taking their average, we can get a combined estimate which is $\sqrt{n}$ times more accurate than each of the individual estimates. Thus, it is reasonable to assume that when we have n users, the amount of effort needed to learn the tool is (approximately) equal to $\dfrac{b}{\sqrt{n}}$ for some constant b.

Resulting dynamics of propagation. As we have mentioned, a person starts learning a new tool if the expected benefit of its use exceeds the cost of learning. Once we have n users, the cost of learning is equal to $\dfrac{b}{\sqrt{n}}$. The expected benefit of leaning the tool is proportional to the average number of problems encountered by the potential user, i.e., to the number x; in other words, this benefit can be described as $a \cdot x$ for some constant a. So, at this stage, only persons for which $a \cdot x \ge \dfrac{b}{\sqrt{n}}$ have an incentive to study this tool. This condition can be described equivalently as $x \ge X_0$, where we denoted $X_0 \stackrel{\text{def}}{=} \dfrac{b}{a \cdot \sqrt{n}}$. According to the power-law distribution, out of N who may be potentially interested in this tool, the total number of persons who have an incentive to study this tool is equal to

$$N \cdot P(X > X_0) = N \cdot C_0 \cdot X_0^{-\alpha} = N \cdot C_0 \cdot \left(\frac{b}{a \cdot \sqrt{n}}\right)^{-\alpha} = c_1 \cdot n^{\alpha/2}, \tag{15}$$

for an appropriate constant $c_1 \stackrel{\text{def}}{=} N \cdot C_0 \cdot \left(\dfrac{b}{a}\right)^{-\alpha}$.

The rate $\dfrac{dn}{dt}$ with which the number of users n increases is proportional to the number of potential users who study the new tool, i.e., to the number of persons who have an incentive to study this tool. Thus, we conclude that

$$\frac{dn}{dt} = c_2 \cdot n^{\alpha/2} \tag{16}$$

for some constant c_2.

Moving terms containing n to the left-hand side and terms containing t to the right-hand side, we get

$$n^{-\alpha/2} \cdot dn = c_2 \cdot dt. \tag{17}$$

Now, we can integrate both sides. The result of this integration depends on the value $\alpha/2$.

When $\alpha/2 = 1$, integration leads to

$$\ln(n) = c_2 \cdot t + C, \tag{18}$$

where C is the integration constant. We want to describe the starting period of idea propagation, when $n(t) = 0$ for $t = 0$. For $n = 0$, however, the left-hand side of (18) is infinite, while the right-hand side is finite. Thus, we cannot have $\alpha/2 = 1$.

When $\alpha/2 \ne 1$, integration of (17) leads to

$$\frac{1}{1-\alpha/2} \cdot n^{1-\alpha/2} = c_2 \cdot t + C. \tag{18}$$

We want to satisfy the requirement that $n(t) = 0$ when $t = 0$. When $t = 0$, the right-hand side of the formula (18) is equal to C. When $n = 0$, the value $n^{1-\alpha/2}$ is equal to 0 when $1-\alpha/2 > 0$ and to ∞ when $1-\alpha/2 < 0$. Since $C < \infty$, the condition that $n(t) = 0$ when $t = 0$ can only be satisfied when $1-\alpha/2 > 0$. In this case, for $t = 0$, the formula (18) takes the form $0 = C$. Substituting $C = 0$ into the formula (18), we conclude that

$$\frac{1}{1-\alpha/2} \cdot n^{1-\alpha/2} = c_2 \cdot t, \tag{19}$$

hence

$$n^{1-\alpha/2} = (c_2 \cdot (1-\alpha/2)) \cdot t. \tag{20}$$

Raising both side by the power $a \stackrel{\text{def}}{=} 1/(1-\alpha/2)$, we conclude that

$$n(t) = c \cdot t^a, \tag{21}$$

where $c \stackrel{\text{def}}{=} (c_2 \cdot (1-\alpha/2))^a$. This is exactly the formula that we wanted to explain.

Conclusion. So, the above light user–heavy users model indeed explains the observed power-law growth of the number of adoptees of a new idea.

Acknowledgments. This work was supported in part by the National Science Foundation grants HRD-0734825 and HRD-1242122 (Cyber-ShARE Center of Excellence), OCI-1135525 (Virtual Learning Commons), and DUE-0926721, by Grants 1 T36 GM078000-01 and 1R43TR000173-01 from the National Institutes of Health, and by a grant N62909-12-1-7039 from the Office of Naval Research.

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