Complete Description of Idempotent Hedges in Fuzzy Logic

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Abstract

In describing expert knowledge, it is often important to properly take into account *hedges* like "very", "somewhat", etc. In particular, fuzzy logic provides a consistent way of describing hedges. For some of the hedges, a repetition changes the meaning: e.g., "very very small" is smaller than "very small". However, other hedges – like "somewhat" – are *idempotent*, in the sense that repeating this hedge twice does not change the meaning. In this paper, we provide a complete description of such idempotent hedges.
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1 Need to Describe Expert Knowledge

In many application areas, experts have knowledge which we would like to use. For example, in medicine, skilled medical professionals can diagnose diseases and assign treatments. We would like to utilize their knowledge, e.g., by formally describing this knowledge to a computer so that other doctors may use this knowledge.

These professionals are willing to share this knowledge, but the rules are described using words from natural knowledge. For example, a dermatologist would say:

- If a blemish is "small", then cut it off.

- If a blemish is "medium", better check again (if we simply cut it, it may leave a scar which is not a good idea).

Such words are often imprecise ("fuzzy"). For the same 1 cm blemish (looking at the previous example) some doctors will consider it small, some will consider it not small, and others will be unsure whether it is small or not.

Fuzzy logic (see, e.g., [1, 2, 3]) is one of the ways to formalize such imprecise words. Specifically, to each such word (like "small") and to each possible value (e.g., "1 cm"), we assign a number from the interval $[0, 1]$ that describes to what extent this value is consistent with this word. If everyone agrees that the blemish is small, we assign 1. If everyone agrees that the blemish is not small, we assign 0. When some experts think it is small and some experts think it is not small, we can assign, e.g., the proportion of experts that think that this is small. For example, if 8 out of 10 experts believe that a 1 cm blemish is small, then we assign a degree of belief of 0.8 to the statement "A 1 cm blemish is small."

2 Hedges in Describing Expert Knowledge

The importance of hedges. Often, experts use "hedges", auxiliary words that slightly modify the meaning of the terms. For example, in addition to the following rule:

- If a patient has high blood pressure, then give him a a medicine.

we may also have the following rules:

- If a person has very high blood pressure, then give him a strong dose of a medicine.

- If a person has somewhat high blood pressure, then give him a weak dose of a medicine.

Representing hedges: a challenge. Ideally, we should elicit, from the experts, their degrees of belief for each hedged property (like "very small", "somewhat small", etc.). However, already eliciting degrees of belief for all possible basic properties (like "small", "medium", etc.) is time-consuming, since there are many possible basic properties. To repeat the same procedure many times for each hedge is not realistic.

Zadeh's idea of how to represent hedges. To avoid this time consuming procedure, Zadeh proposed a simplified way to elicit such degrees.

This simplified way is based on the following idea. If we believe with degree 0.9 that something is high, then it is reasonable to expect that our degree of belief that the same value is very high should be smaller than 0.9, while our degree of belief that this value is somewhat high should be larger than 0.9. Based on this idea Zadeh proposed to describe each hedge as a function $f(x)$ that inputs the degree of belief x in the original statement and outputs the degree of belief $f(x)$ in the correspondingly hedged statement. For example, if for a function $f(x)$ corresponding to "very," we have $f(0.9) = 0.8$, this means that if we believe with degree 0.9 that a blood pressure of 160 is high, then we estimate our degree of belief that this blood pressure is very high as 0.8.

Formalizing Zadeh's idea. Since degrees are numbers from the interval $[0, 1]$, a hedge can therefore be described as a function from this interval to itself.

If our degree of belief in a statement increases, then our degree of belief in the hedged statement should also increase or at least not decrease. For example, if our degree of belief that blood pressure is high increases, then our degree of belief that blood pressure is very high should also increase. In terms of the function f describing the hedge this means that if x is smaller than x' then $f(x)$ should be smaller than or equal to $f(x')$. In other words, the function corresponding to the hedge should be *non-decreasing*.

Similarly, if our belief in the original statement changes very little, it is reasonable to expect that our degree in the hedged statement should also not change too much. In other words, it is reasonable to require that the function $f(x)$ describing the hedge is *continuous*.

So, we arrive at the formal definition of a hedge.

Definition 1. *By a* hedge *we mean a continuous, non-decreasing function from* $[0, 1]$ *to* $[0, 1]$.

3   What are Idempotent Hedges and Why Are They Important?

Since linguistic hedges are important, it is necessary to describe the functions corresponding to different linguistic hedges. In general, a linguistic hedge can be represented any continuous non-decreasing function. However, some linguistic hedges have additional properties. It is desirable to see whether these properties can simplify the description of these hedges.

One of such properties is related to the possibility to apply a hedge to an already hedged statement. For example, we can apply the hedge "very" to the hedged statement "very small", and get a new statement "very very small". For some hedges, if you apply the same hedge to the hedged statement, it changes the meaning. For example, "very very small" is stronger than simply "very small".

In terms of a function $f(x)$ describing the hedge "very", if we believe that some value is small with degree x, then our degree of belief that this value is very small is $x' = f(x)$, and our degree of belief that this value is very very small is $f(x') = f(f(x))$. In these terms, for a hedge like "very", for which "very very" is different from "very", the value $f(f(x))$ is different from $f(x)$.

However, for some hedges, the second application does not change the meaning. For example, if we say that a statement is to some extent true or if we say that to some extent we believe that a statement is to some extent true, then the second phrase has, in effect, the same meaning as the first one, except in a clumsier way. There may be a subtle difference between the meanings, but nothing as radical as the difference between "very small" and "very very small".

Another example of such a hedge is "somewhat." While "very very small" is different from "very small", "somewhat somewhat small" looks like having the same meaning as "somewhat small". For such hedges, the degree $f(f(x))$ is the same as the degree $f(x)$.

In mathematics, operations for which $f(f(x))$ is equal to $f(x)$ are called *idempotent*. Therefore, it is reasonable to call hedges that satisfy this equality *idempotent hedges*.

Definition 2. *A hedge $f(x)$ is called* idempotent *if $f(f(x)) = f(x)$ for all x.*

Formulation of the problem. Since experts use idempotent hedges to describe their knowledge, it is desirable to have a complete description of all such hedges.

4   Main Result: Complete Description of All Idempotent Hedges

Proposition. *Every idempotent hedge $f(x)$ has the following form:*

- *$f(x) = a$ if $x \leq a$,*

- *$f(x) = x$ if $a \leq x \leq b$, and*

- *$f(x) = b$ if $x \geq b$.*

Vice versa, each function $f(x)$ of this form is an idempotent hedge.

Proof: First, due to idempotence property, $f(t) = t$ for all t for which $f(x) = t$ for some x.

Then, since f is monotonic, $f(0) \leq f(x) \leq f(1)$. Since f is continuous, every t from $f(0)$ to $f(1)$ can be represented as $f(x)$ for some x.

This takes care of $f(t)$ for t from $a = f(0)$ to $b = f(1)$.

Let us now prove that $f(t) = a$ for $0 \leq t \leq a$. Indeed, for such t, monotonicity implies that $f(0) \leq f(t) \leq f(a)$. Here, by definition of a, we have $f(0) = a$, and idempotence implies that $f(f(0)) = f(0)$, i.e., that $f(a) = a$. Thus, the inequality $f(0) \leq f(t) \leq f(a)$ takes the form $a \leq f(t) \leq a$, hence $f(t) = a$.

Similarly, for $b \leq t \leq 1$, monotonicity implies that $f(b) \leq f(t) \leq f(1)$. Here, by definition of b, we have $f(1) = b$, and idempotence implies that $f(f(1)) = f(1)$, i.e., that $f(b) = b$. Thus, the inequality $f(b) \leq f(t) \leq f(1)$ takes the form $b \leq f(t) \leq b$, hence $f(t) = b$.

It is also easy to check that each function of the form described in the proposition is an idempotent hedge. The proposition is proven.

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