A Simple Geometric Model Provides a Possible Quantitative Explanation of the Advantages of Immediate Feedback in Student Learning

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Abstract

Calculus is a known bottleneck for many students studying science and engineering. Various techniques have been developed to enhance the students' success. A recent study [1] published in the *Notices of American Mathematical Society* showed that only one factor determines the success of a technique: the presence of immediate feedback. On average, students who receive immediate feedback learn twice faster than students who are taught in a more traditional way, with a serious feedback only once or twice a semester (after a test).

The very fact that immediate feedback is helpful is not surprising: it helps the student clear misconceptions and avoid the wrong paths. However, the fact that different techniques involving feedback lead to practically the same learning speed-up is intriguing. To explain this speed-up, we provide a simplified first-order description of a learning process in simple geometric terms. We show that already in this first approximation, the geometric description leads to the observed two-fold speed-up in learning.

1 Immediate Feedback Improves Student Learning: Empirical Data

Student understanding is extremely important. One of the main objectives of a course – whether it is calculus or physics or any other course – is to enable students to understand the main concepts of this course. Of course, it is also desirable that the students learn the corresponding methods and algorithms, but understanding is the primary goal. If a student does not remember by heart how to compute the derivative of a product, he or she can look up the formula on the web or even derive the formula – and so, most probably, this student will succeed in the following classes which depend on the use of derivatives. However, if a student does not have a good understanding of what is a derivative, then even if this student remembers some formulas, the student will probably not be able to decide which formula to apply in what situation.

How to gauge student understanding. To properly gauge student's understanding, several disciplines have developed *concept inventories*, sets of important basic concepts and questions testing the students' understanding of these concepts. The first such concept inventory was developed in physics, to gauge the students' understanding of the basic concepts of Newtonian mechanics such as the concept of force; the corresponding Force Concept Inventory (FCI) is described in [4, 5, 6, 7, 8]. A similar Calculus Concept Inventory (CCI) is described in [2, 3].

A student's degree of understanding is measured by the percentage of the questions that are answered correctly. The class's degree of understanding is measured by averaging the students' degrees. An ideal situation is when everyone has a perfect 100% understanding; in this case, the average score is 100%. In practice, the average score is smaller than 100%.

How to compare different teaching techniques. To gauge how successful is a given teaching technique, we can measure the average score μ_0 before the class and the average score μ_f after the class. A perfect class is when the whole difference $100 - \mu_0$ disappeared, i.e., the students' average score went from μ_0 to $\mu_f = 100$. In practice, of course, the students' gain $\mu_f - \mu_0$ is somewhat smaller than the ideal gain $100 - \mu_0$. It is reasonable to measure the success of a teaching method by describing which portion of the ideal gain is covered, i.e., by the ratio

$$g \stackrel{\text{def}}{=} \frac{\mu_f - \mu_0}{100 - \mu_0}.$$

Empirical results. It turns out that for different teaching methods, the normalized gain g does not depend on the initial level μ_0, does not depend on the textbook used or on the teacher. Only one factor determines the value g: the absence or presence of immediate feedback.

In the traditionally taught classes, where the students get their major feedback only after their first midterm exam, the normalized gain g is consistently smaller than in the classes where the students got immediate feedback during every class period.

Specifically, for traditionally taught classes, the average value of the gain is $g \approx 0.23$, while for the classes with an immediate feedback, the average value of the gain is $g \approx 0.48$; see, e.g., [1, 4].

In other words, students who receive immediate feedback, on average, learn twice faster than students who are taught by traditional methods.

Natural question. The consistent appearance of the doubling of the rate of learning seems to indicate that there is a fundamental reason behind this empirical result.

What we do in this paper. In this paper, we provide a possible geometric explanation for the above empirical result.

2 A Simplified First-Order Description of Learning in Geometric Terms

Why geometry. Learning means that the student – who did not originally know the material – becomes knowledgeable of this material. To check how well a student learned, we can apply different tests. Based on the results of these tests, we can determine the current state of the student knowledge. In other words, at any given moment of time, the state of the student's knowledge can be characterized by several numbers $(x_1, \ldots, x_n)$ – the student's scores on different parts of the test.

Each such state can be naturally represented as a point in the n-dimensional space – namely, a point with coordinates $x_1, \ldots, x_n$. In the starting state S, the student does not know the material; the desired state D describes the situation when a student has the desired knowledge. When a student learns, the student's state of knowledge changes continuously forming a (continuous) trajectory γ which starts at the starting state S and ends up at the desired state D.

First simplifying assumption: all students learn at the same rate. Some students learn faster, others learn slower. The above empirical fact is not about their individual learning rates, it is about the *average* rates of student learning, averaged over all kinds of students. From this viewpoint, it makes sense to simplify the complex actual situation – in which different students have different learning rates – with a simplified model, in which all the students have the same average learning rate.

Let us give an example of why such a replacement makes sense when we only consider averages:

- if we are want to study the difference between people's appetites, it makes sense to keep their differing heights intact;
- however, if we are planning to serve a meal to a large group of people, it makes perfect sense, when ordering food ingredients, to ignore the individual differences and assume that everyone has an average appetite.

In geometric terms, the rate of learning corresponds to the rate with which the student's state changes, i.e., corresponds to how far the student's state of knowledge changes in a given period of time. In these terms, the assumption that all the students have the same learning rate means that the states corresponding to different students change with the same rate. In other words, in this geometric model, the time that it takes for a student to get from the initial state S to the desired state D is proportional to the total length of the corresponding curve γ.

In these terms, to explain the fact that students who receive instant interaction learn twice faster means that on average, we need to show that their learning trajectories are, on average, twice shorter.

Second simplifying assumption: the shape of the learning trajectories. In the beginning, a student may be eager to study, but often, he/she is not sure which direction to go. A student usually has misconceptions about physics and/or calculus, misconception that may lead the student in a wrong direction. We can describe this by assuming that when a student starts at the starting point S, he/she moves in a random direction.

In situations when the student deviated from the direction towards the desired state D, a feedback enables the student understand that he/she is going in the wrong direction. After the feedback, the student corrects his/her trajectory.

In the case of immediate feedback, this correction comes right away, so, in effect, the student immediately starts following the right direction. In other words, in learning with immediate feedback, the student's learning trajectory is a straight line which goes directly from S to D.

In the traditional learning, feedback comes only with midterm exams. Usually, there are two midterm exams, and they are scheduled in such a way that between themselves, they cover all the material studied in the course, and each covers approximately the same amount of material. Thus, the first midterm exam usually covers half of the material. In geometric terms, it means that this exam is given once the student covered half of the distance between S and D. This exam checks whether the student has correctly reached the midpoint $M \stackrel{\text{def}}{=} \dfrac{S+D}{2}$ between S and D. Once the student has covered the half-distance $d/2$ in the originally selected direction, the results of the first midterm exam provides a necessary correction, and the student starts going straight towards the correct midpoint M. After that, the same process starts again: the student goes for $d/2$ in the random direction, and then comes back to D.

Resulting geometric description of leaning with and without immediate feedback. In learning with immediate feedback, a student follows a straight line from S to D. The length of the corresponding trajectory is equal to the distance $d \stackrel{\text{def}}{=} \rho(S, D)$ between the states S and D.

In learning without immediate feedback, a student first follows a straight line of length $d/2$ which goes in a random direction, then goes straight to the midpoint M, then again follows a straight line of length $d/2$ in a random direction, and finally takes a straight like to D.

Third simplifying assumption: the state space is 1-D. While in general, we can think of different numerical characteristics describing different aspects of student knowledge, in practice, we are pretty comfortable using a single number – usually, an overall grade for the course – to characterize the student's state of knowledge. It is therefore reasonable to make one more simplifying assumption: that the state of a student is characterized by only one parameter x_1.

Let us compare the lengths of the corresponding trajectories. Under our simplifying assumption, the learning time is proportional to the length of the corresponding trajectory. Thus, to compare the learning rates, we need to compare the lengths of the corresponding trajectories.

In case of immediate feedback, the learning trajectory has length d. So, to make a comparison, we must estimate the length of a trajectory corresponding to the traditional learning.

This trajectory consists of two similar parts: the part connecting S and M and the part connecting M and D. Hence, to estimate the total average length, it is sufficient to estimate the average length from S to M and then multiply the result by two.

3 Analysis of the Model and the Resulting Explanation

Analysis. In case of immediate feedback, the learning trajectory has length d.

In the case of traditional learning, under the 1-D assumption, a student initially goes either in the correct direction or in the opposite (wrong) direction; the idea that the direction is chosen randomly can be naturally formalized as an assumption that both directions occur with equal probability 1/2.

If the student's trajectory initially moves in the correct direction, then after traveling the distance $d/2$, the state gets exactly into the desired midpoint D; so, the overall length of the S-to-M of the trajectory is exactly $d/2$.

If the student's trajectory initially goes in the wrong direction, then the student ends up at a point at the same distance $d/2$ from S but on the wrong side of S. Getting back to M then means first going back to S and then going from S to M. The overall length of this trajectory is thus $3d/2$.

With probability 1/2, the length is $d/2$, with probability 1/2, the length is $3d/2$. So, the average length of the S-to-M part of the learning trajectory is equal to

$$\frac{1}{2} \cdot \frac{d}{2} + \frac{1}{2} \cdot \frac{3d}{2} = d.$$

The average length of the whole trajectory is double that, i.e., $2d$ – twice larger than the length corresponding to immediate feedback.

Since we assumed that the learning time is proportional to the length of the learning trajectory, we can thus make the following conclusion.

Conclusion. In this 1-D model, a student following an instant feedback trajectory reaches the desired state, on average twice faster than a student following the traditional-learning trajectory. This is exactly what we wanted to explain.

Acknowledgments. This work was supported in part by the National Science Foundation grants HRD-0734825 and HRD-1242122 (Cyber-ShARE Center of Excellence) and DUE-0926721.

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