A FEASIBLE ALGORITHM FOR CHECKING n-SCISSORS CONGRUENCE OF POLYHEDRA IN $\mathbb{R}^d$

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Abstract

While in $\mathbb{R}^2$, every two polygons of the same area are *scissors congruent* (i.e., they can be both decomposed into the same finite number of pair-wise congruent polygonal pieces), in $\mathbb{R}^3$, there are polyhedra P and P' of the same volume which are not scissors-congruent. It is therefore necessary, given two polyhedra, to check whether they are scissors-congruent (and if yes – to find the corresponding decompositions). It is known that while there are algorithms for performing this checking-and-finding task, no such algorithm can be feasible – their worst-case computation time grows (at least) exponentially, so even for reasonable size inputs, the computation time exceeds the lifetime of the Universe. It is therefore desirable to find cases when feasible algorithms are possible.

In this paper, we show that for each dimension d, a feasible algorithm is possible if we fix some integer n and look for *n-scissors-congruence* in $\mathbb{R}^d$ – i.e., for possibility to represent P and P' as a union of $\le n$ *simplexes*.

1 Formulation of the Problem

Scissors congruence: brief reminder. In a plane, every two polygons P and P' of equal area $A(P) = A(P')$ are *scissors congruent* (*equidecomposable*) – i.e., they can be both decomposed into the same finite number of pair-wise congruent polygonal pieces: $P = P_1 \cup \ldots \cup P_p$, $P' = P'_1 \cup \ldots \cup P'_p$, and $P_i \sim P'_i$.

In one of the 23 problems that D. Hilbert formulated in 1900 as a challenge to the 20 century mathematics, namely, in Problem No. 3, Hilbert asked whether every two polyhedra P and P' with the same volume $V(P) = V(P')$ are scissors congruent[8]. This problem was the first to be solved: already in 1900, Dehn proved [3, 4] that there exist a tetrahedron of volume 1 which is not scissors congruent with a unit cube; see, e.g., [1, 5, 6, 12] for a detailed overview.

Algorithm for checking scissors congruence. Let us consider polyhedra which can be constructed by geometric constructions. It is well known that for such polyhedra, all vertices have *algebraic* coordinates (i.e., values which are roots of polynomials with integer coefficients); see, e.g., [2].

In [10], we described an algorithm for checking whether two polyhedra with algebraic coordinates in $\mathbb{R}^3$ (or in $\mathbb{R}^4$) are scissor congruent. When the polyhedra are scissor congruent, this algorithm also enables us to find the corresponding scissor decomposition P_i and P'_i.

In general, the task of checking scissors congruence and – if yes – finding the corresponding decompositions is not feasible. In [9], we have shown that in general, the problem of constructing the corresponding scissor decomposition requires computation time t which grows exponentially with the size s of the input: $t \geq c^s$ for some $c > 1$.

In theoretical computer science, such algorithms are known as *not feasible*, since already for reasonable sizes s, the time c^s exceeds the lifetime of the Universe – and thus, it is not possible to perform these computations. Only algorithms whose computation time is bounded by a polynomial of the size s of the input are considered ro be *feasible*; see, e.g., [11, 13].

A natural question. Since the general problem is not feasible, it is desirable to find feasible cases.

What we do in this paper. In this paper, we show that for every number n, there is a feasible algorithm for checking n*-scissors congruence* – – possibility to represent P and P' as a union of $\leq n$ *simplexes*.

2 Main Result

Notion of n-scissors congruence. Each polygon can be decomposed into triangles; each 3-D polyhedron can be decomposed into tetrahedra; in general, each polyhedron in $\mathbb{R}^d$ can be decomposed into simplexes.

So, P and P' are scissors congruent if and only if they can de decomposed into mutually congruent simplexes.

Let us say that P and P' are *n-scissors congruent* if they can be both decomposed into $\le n$ pair-wise congruent simplexes.

Towards the main result. Let us fix the dimension d, the number of simplexes n, and a coordinate system in $\mathbb{R}^d$.

A simplex in a d-dimensional space can be described by the coordinates of its $d+1$ vertices, i.e., by $d \cdot (d+1)$ real numbers. To describe all $\le n$ simplexes forming P and all $\le n$ simplexes forming P', we thus need $\le N \stackrel{\text{def}}{=} 2n \cdot d \cdot (d+1)$ real numbers; let us denote these numbers by $x_1, \ldots, x_N$.

In these terms, the n-scissors congruence of two polyhedra P and P' can be described as

$$\exists x_1 \ldots \exists x_N \, d(x_1, \ldots, x_N), \tag{1}$$

where $d(x_1, \ldots, x_N)$ means that the corresponding simplexes form P and P' and that the simplexes are mutually congruent.

Two simplexes are congruent if and only if the corresponding distances are equal. The equality of the distances is equivalent to equality of the squares of these distances – and, by definition of the Euclidean distance, these squares are quadratic functions of the coordinates. Thus, pairwise congruence is equivalent to $(d_1 = 0) \,\&\, (d_2 = 0) \,\&\, \ldots$ for appropriate polynomials d_1, d_2, $\ldots$

The fact that P is composed of the corresponding simplexes can also be described in terms of similar equalities and inequalities. So, n-scissors congruence is equivalent to a formula of the type (1), where $d(x_1, \ldots, x_N)$ is a propositional combination of formulas of the type $d_i = 0$ and $d_i \ge 0$, where d_i are polynomials.

Once the number of variables N is fixed, there is an feasible algorithm that checks the validity of such formulas; see, e.g., [7].

For the case when the corresponding formula is valid, this algorithm also finds the values x_i which satisfy this formula.

Conclusion. For each dimension d and number of simplexes n, by applying the algorithm from [7] to the formula (1), we get a feasible algorithm which:

- checks whether two given polyhedra P and P' are n-scissors congruent, and
- if they are, produced the corresponding decomposition.

Discussion. Similar feasible algorithms are possible:

- for checking n-scissor congruence of polyhedra is a d-dimensional sphere;
- for situations when we consider only congruence modulo shift;
- when we look for the possibility to *add* $\leq n$ mutually congruent simplexes to P and P' so that the "padded" polyhedra are n-scissor congruent,

and in many other related problems described in [1].

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References

[1] V. G. Boltianskii, *Hilbert's Third Problem*, V. H. Winston & Sons, Washington, D.C., 1978.

[2] R. Courant and H. Robbins, "Geometric constructions. The algebra of number fields," Ch. 3 in *What Is Mathematics?: An Elementary Approach to Ideas and Methods*, Oxford University Press, Oxford, England, UK, 1996, pp. 117–164.

[3] M. Dehn, "Über raumgleiche Polyeder", *Nachrichten von der Königliche Gesellschaft der Wissenschaften zu Göttingen, Mathematik-Physik Klasse*, 1900, pp. 345–354.

[4] M. Dehn, "Über den Rauminhalt", *Mathematische Annalen*, 1901, No. 3, pp. 465–478.

[5] J. L. Dupont, *Scissors Congruences, Group Homology and Characteristic Classes*, World Scientific, Singapore, 2001.

[6] J. L. Dupont, "What is ... a scissors congruence?", *Notices of the American Mathematical Society*, 2012, Vol. 59, No. 9, pp. 1242–1244.

[7] D. Yu. Grigor'ev and N. N. Vorobjov (Jr.), "Solving systems of polynomial inequalities in subexponential time", *Journal of Symbolic Computation*, 1988, Vol. 5, No. 1/2, pp. 37–64.

[8] D. Hilbert, "Mathematische Probleme", *Nachrichten von der Königliche Gesellschaft der Wissenschaften zu Göttingen*, 1900, pp. 253–297; English translation: "Mathematical Problems, lecture delivered before the International Congress of Mathematics in Paris in 1900", translated in *Bulletin of the American Mathematical Society*, 1902, Vol. 8, pp. 437–479; reprinted in F. E. Browder (ed.), *Mathematical Developments Arising from Hilbert's Problems*, American Mathematical Society, Providence, Rhode Island, 1976.

[9] O. Kosheleva and V. Kreinovich. "Geombinatorics, computational complexity, and saving environment: let's start", *Geombinatorics*, 1994, Vol. 3, No. 3, pp. 90–99.

[10] V. Kreinovich, "Equidecomposability (scissors congruence) of polyhedra in R^3 and R^4 is algorithmically decidable: Hilbert's 3rd Problem revisited", *Geombinatorics*, 2008, Vol. 18, No. 1, pp. 26–34.

[11] V. Kreinovich, A. Lakeyev, J. Rohn, and P. Kahl, *Computational Complexity and Feasibility of Data Processing and Interval Computations*, Kluwer, Dordrecht, 1998.

[12] W. D. Neumann, "Hilbert's 3rd Problem and invariants of 3-manifolds", In: I. Rivin, C. Rourke, and C. Series (eds.), *The David Epstein 60th Birthday Festschrift*, Geometry & Topology Monographs Series, Vol. 1, Mathematical Sciences Publishers, Berkeley, California, 1998, pp. 383–411.

[13] C. Papadimitriou, *Computational Complexity*, Addison Welsey, Reading, Massachusetts, 1994.