Fuzzy, Intuitionistic Fuzzy, What Next?

Vladik Kreinovich[1] and Bui Cong Cuong[2]

[1] University of Texas at El Paso, El Paso, TX 79968, USA
vladik@utep.edu
[2] Institute of Mathematics, Vietnam Academy of Science and Technology, 18 Hoang Quoc Viet, Cau Giay, Hanoi, Vietnam
bccuong@gmail.com

Abstract. In the 1980s, Krassimir Atanassov proposed an important generalization of fuzzy sets, fuzzy logic, and fuzzy techniques – intuitionistic fuzzy approach, which provides a more accurate description of expert knowledge. In this paper, we describe a natural way how the main ideas behind the intuitionistic fuzzy approach can be expanded even further, towards an even more accurate description of experts' knowledge.

1 Fuzzy Logic: A Brief Reminder

The main objective of this paper is to describe the main ideas behind intuitionistic fuzzy logic and to describe how these ideas can be expanded. To do that, we need recall the main motivations and the main ideas behind the original fuzzy logic; for details, see, e.g., [6, 10, 11].

It is important to describe and process expert knowledge. In many practical situations, from medicine to driving to military planning to decisions on whether to accept a paper for publication, we rely on expert opinions.

In every field, there are a few top experts. For example, in every medical area, there are top specialists in this area. In the ideal world, every patient in need of a surgery would be operated by the world's top surgeon, and every person would get an advice from the world's top financial advisor on how to invest his or her savings. Since it is not possible for a few top surgeons to perform all the operations and for top financial advisors to advice everyone, it is desirable to design computer-based system which would incorporate the advice of the top experts – and thus help other experts provide a better quality advice. Such computer-based systems are often called *expert systems*.

Experts can describe their knowledge in terms of statements S_1, S_2, ... (e.g., "if the P/E ratio of a stock goes above a certain threshold t_0, it is recommended to sell it"). In some situations, when we have a query Q – e.g., whether to sell a given stock – we can use one of the expert rules. In many other cases, however, none of the expert rules can lead directly to the desired answer, but a proper combination of the rules can help. For example, in medical expert systems, we rarely have a rule directly linking patient's symptoms with the appropriate treatment, but we have rules which link symptoms with diseases, and we have rules which link diseases with treatments. By combining the corresponding rules, we can get an answer to the query. The part of an expert system which, given a query, tries to deduce the corresponding statement or its negation from the expert rules, is known as an *inference engine*.

Uncertainty of expert knowledge. In using expert knowledge, we need to take into account that experts are usually not 100% confident that their statements are universally valid. For example, if a patient sneezes and coughs, a medical doctor will conclude that it is most probably cold, flu, or allergy, but the doctor also understands that there is a possibility of some rarer situations with similar symptoms.

A natural way to gauge the experts' uncertainty is to ask the experts to mark their uncertainty on a scale from 0 to some integer n (e.g., on a scale from 0 to 5), so that 0 corresponds to no certainty at all, and n correspond to the absolute certainty. If an expert marks m on a scale from 0 to n, then we claim that the expert's degree of certainty in his/her statement is the ratio m/n.

How to process experts' uncertainty: towards a precise formulation of the problem. Since the experts are not 100% sure in their statements, we are therefore not sure about the expert system's conclusion either. It is therefore important to make sure that the expert system not only provides a "yes" or "no" (or more complex) answer to a given query, but that the user will also get a degree with which we are confident in this answer.

For example, if a medical expert system recommends a surgery, and the resulting confidence is 99%, then it is probably a good idea to undergo this surgery. However, if the resulting degree of confidence that this answer is correct is about 50%, maybe it is better to perform some additional tests so that we may become clearer on the diagnosis.

It is thus important, once we have derived a statement Q from the expert knowledge base $\{S_1, S_2, \ldots\}$, to provide the user with the degree $d(Q)$ that the resulting statement Q is correct. In some cases, there is only one chain of reasoning leading to the conclusion Q, and this chain involves statements $S_{i_1}, \ldots, S_{i_k}$. In this case, all these statements need to be true for Q to be true: if one of the statements in the chain is false, then the whole chain of reasoning collapses. In these cases, Q is true if the statement $S_{i_1} \,\&\, \ldots \,\&\, S_{i_k}$ is true. Thus, to gauge our degree of belief in Q, we must be able to estimate the degree of belief in a statement $S_{i_1} \,\&\, \ldots \,\&\, S_{i_k}$.

In general, we may have several derivations of Q – e.g., we may have several different observations supporting the same diagnosis. In this case, Q is deduced if at least one of the corresponding derivation chains is valid, i.e., if a propositional formula of the following type holds:

$$(S_{i_1} \,\&\, \ldots \,\&\, S_{i_k}) \vee \left(S_{i'_1} \,\&\, \ldots \,\&\, S_{i'_{k'}}\right) \vee \ldots$$

Approximate estimation is needed. In other words, we would like to estimate the degree of belief in different propositional combinations of the original statements S_i. Of course, if we only know the expert's degrees of belief $d(S_1)$ and $d(S_2)$ of different statements S_1 and S_2, we cannot uniquely determine the expert's degree of certainty $d(S_1 \,\&\, S_2)$. For example, if S_1 means that a fair coin falls heads, and $S_2 = S_1$, then it is reasonable to take $d(S_1) = d(S_2) = 0.5$ and, thus, $d(S_1 \,\&\, S_2) = d(S_1) = 0.5$. On the other hand, if we take $S_2 = \neg S_1$, then still $d(S_1) = d(S_2) = 0.5$ but now $d(S_1 \,\&\, S_2) = 0$.

Since we cannot uniquely determine the degrees of certainty in all possible propositional combinations based only on the degrees $d(S_i)$, ideally, we should also find the degrees of certainty in all these propositional combinations. The problem is that for N original statements, we need $> 2^N$ different degrees to describe, e.g., the degrees of certainty in different combinations $S_{i_1} \,\&\, \ldots\, S_{i_n}$ ($> 2^N$ because we have $2^N - 1$ possible non-empty subsets $\{i_1, \ldots, i_n\} \subseteq \{1, \ldots, N\}$).

Even for middle-size $N \approx 100$, the value 2^N is astronomically high. It is not possible to elicit all these degrees of certainty from the expert. Thus, no matter how much information we elicit, we will always have propositional combinations for which we do not know the corresponding degrees, combinations for which these degrees must be estimated.

How to estimate the corresponding degrees: fuzzy-motivated idea of negation-, "and"- and "or"-operations. A general propositional combination is obtained from the original statement by using the logical connectives $\neg$ ("not"), & ("and"), $\vee$ ("or"). Since we do not know the degrees of all composite statements, we inevitable face the following problem:

- for some statements A and B, we know the expert's degrees of certainty $d(A)$ and $d(B)$ in these statements;
- we need to estimate the expert's degree of certainty in the statements $\neg A$, $A \,\&\, B$ and/or $A \vee B$.

Negation operations. In this situation, to come up with the desired estimate $d(\neg A)$, the only information that we can use consists of a single number $d(A)$. Let us denote the estimate for $d(\neg A)$ corresponding to the given value $d(A)$ by $f_\neg(d(A))$. The corresponding function is usually known as an *negation operation*.

How can we choose this negation operation? Let us first describe some reasonable properties that this function should satisfy. First, we can take into account that $\neg(\neg A)$ usually means the same as A. By applying the negation operation $f_\neg$ to the estimated degree of certainty $d(\neg A) \approx f_\neg(d(A))$, we can estimate the expert's degree of certainty in $\neg(\neg A))$ as $f_\neg(f_\neg(d(A))$. It is reasonable to require that this estimate coincide with the original value $d(A)$: $f_\neg(f_\neg(d(A)) = d(A)$. This equality must hold for all possible values $a = d(A) \in [0, 1]$, so we must have $f_\neg(f_\neg(a)) = a$ for all a. In mathematical terms, this means that the function $f_\neg(a)$ is an *involution*.

When A is absolutely false, and $d(A) = 0$, then $\neg A$ should be absolutely true, i.e., we should have $f_\&(0) = 1$. Similarly, if A is absolute true and $d(A) = 1$, then $\neg A$ should be absolutely false, i.e., we should have $f_\&(1) = 0$. In general, the more we believe in A, the less we should believe in $\neg A$, so the function $f_\&(a)$ must be decreasing.

The most widely used negation operation is $f_{\&}(a) = 1 - a$, it satisfies all these properties; there are also other negation operations which are sometimes used in fuzzy systems.

"And"-operations. To come up with the desired estimate $d(A \,\&\, B)$, the only information that we can use consists of two numbers $d(A)$ and $d(B)$. Let us denote the estimate for $d(A \,\&\, B)$ corresponding to the given values $d(A)$ and $d(B)$ by $f_{\&}(d(A), d(B))$. The corresponding function is usually known as an *"and"-operation*, or *t-norm*.

How can we choose the "and"-operation? Let us first describe some reasonable properties that the corresponding function $f_{\&}(a, b)$ should satisfy. First, since $A \,\&\, B$ means the same as $B \,\&\, A$, it is reasonable to require that the two estimates $f_{\&}(d(A), d(B))$ and $f_{\&}(d(B), d(A))$ corresponding to different orders of A and B should be the same. This must be true for all possible values of $a = d(A)$ and $b = d(B)$; this means that we must have $f_{\&}(a, b) = f_{\&}(b, a)$ for all real values $a, b \in [0, 1]$. In other words, an "and"-operation must be *commutative.*

Similarly, $A \,\&\, (B \,\&\, C)$ means the same as $(A \,\&\, B) \,\&\, C$. If we follow the first expression, then, to estimate the corresponding degree of certainty, we first estimate $d(A \,\&\, B)$ as $f_{\&}(d(A), d(B))$ and then use the "and"-operation to combine this estimate and the degree of certainty $d(C)$ into an estimate $f_{\&}(f_{\&}(d(A), d(B)), d(C))$. Alternatively, if we follow the second expression, we end up with the estimate $f_{\&}(d(A), f_{\&}(d(B), d(C))$. It is reasonable to require that, since $A \,\&\, (B \,\&\, C) \equiv (A \,\&\, B) \,\&\, C$, these two estimates should coincide, i.e., that the "and"-operation be associative.

The expert's degree of confidence $d(A \,\&\, B)$ that both A and B are true should not exceed the degree of confidence that A is true. Thus, we should have $d(A \,\&\, B) \le d(A)$. It is therefore reasonable to require that $f_{\&}(a, b) \le a$ – and thus, that $f_{\&}(0, a) = 0$ for all a.

It is also reasonable to require that when $d(A) = 1$ (i.e., when we are 100% certain in A), then we should have $A \,\&\, B$ equivalent to B, so $f_{\&}(1, b) = b$ for all b. If we increase our degree of confidence in A and/or B, this should not lead to a decrease in our confidence in $A \,\&\, B$; this means that the "and"-operation should be monotonic: $a \le a'$ and $b \le b'$ implies $f_{\&}(a, b) \le f_{\&}(a', b')$. Finally, small changes in $d(A)$ and $d(B)$ should not lead to a drastic change in $d(A \,\&\, B)$, so the "and"-operation must be continuous.

"Or"-operations. Similarly, if we denote by $f_{\vee}(d(A), d(B))$ the estimate for $d(A \vee B)$, then the corresponding "or"-operation (also known, for historical reasons, as *t-conorm*) must be commutative, associative, monotonic, continuous, and satisfy the properties $f_{\vee}(0, a) = a$ and $f_{\vee}(1, a) = 1$ for all a.

Selecting different propositional operations: an empirical task. There are many different negation, "and"-, and "or"-operations which satisfy these properties; for each application area, we select the operations which best describe the reasoning of experts in this area, i.e., for which the resulting estimates for the expert's degrees of confidence in composite statement are the closest to the estimates for $d(\neg A)$, $d(A \,\&\, B)$, and $d(A \vee B)$ produced by the experts.

This idea was first implemented for the world's first expert system MYCIN – Stanford's expert system for diagnosing rare blood diseases; see, e.g., [3]. The authors of MYCIN tried different possible "and" and "or-operations and found the one which was the best fit for the actual reasoning of medical experts. It is worth mentioning that when they tried to apply their expert system to a different application area – geophysics – it turned out that the medical-generated "and"- and "or"-operations did not lead to good results, different operations had to be used.

Common misunderstanding. The reason why in fuzzy techniques (and in expert systems in general), we estimate the degree of confidence $d(A\,\&\,B)$ by applying an "and"-operation to $d(A)$ and $d(B)$ is *not* because we are under an illusion that the expert's degree of confidence in $A\,\&\,B$ is uniquely determined by his/her degrees of confidence in A and B. Everyone understands that there is no uniqueness here, the above example of a coin falling heads or tails is clear. What the "and"-operation produces is an *approximation* to the actual expert's degree of belief in $A\,\&\,B$.

We do not use this approximation because we are under some erroneous belief that "and"- and "or"-operations are truth-functional, but simply because we cannot realistically elicit all the degrees of confidence in all the propositional combinations from all the experts, and we therefore need to estimate the unknown degrees of certainty based on the known ones.

2 From Fuzzy to Intuitionistic Fuzzy

How can we improve the traditional fuzzy approach? One of the main ideas behind the traditional fuzzy approach is that, since we cannot elicit the expert's degrees of confidence in all possible propositional combinations of their original statements $S_1, \ldots, S_n$, we:

– extract the degrees of confidence $d(S_i)$ in these statements, and then
– use negation, "and"-, and "or"-operations to estimate the expert's degrees of belief in different propositional combinations.

To make these estimates more accurate, a natural idea is to extract, from the expert, not just his/her degrees of confidence in the original statements, but also degrees of confidence in some propositional combinations of these statements – at least the simplest ones.

This idea naturally leads to intuitionistic fuzzy logic. Which propositional combinations are the simplest? The more original statements are involved in a combination, the more propositional connectives are used, the more complex the statements. From this viewpoint, the simplest propositional combinations are the ones which has the smallest number of the original statements – one – combined by the smallest possible number of possible connectives: one. There are three possible connectives: negation, "and", and "or". "And" and "or" requires at least two original statements to combine (since $S_i\,\&\,S_i$ and $S_i \vee S_i$ mean the same as S_i). So, the only way to have a single original statement is by using negation. Thus, the simplest possible propositional combinations are negations $\neg S_i$.

Thus, to come up with a more adequate description of expert's degree of certainty, a natural next step is not only to elicit the expert's degrees of confidence $d(S_i)$ in their original statements, but also their degrees of confidence $d(\neg S_i)$ in their negations. In other words, to describe the expert's certainty about his/her statement S_i, instead of a single number $d(S_i)$, we now use a *pair* of numbers $d(S_i)$ and $d(\neg S_i)$. This is, in a nutshell, the main idea behind Atanassov's intuitionistic fuzzy logic; see, e.g., [1, 2].

This idea makes perfect sense. Intuitively, the above idea makes perfect sense. In contrast to the traditional fuzzy logic, this idea enables us to distinguish between two different situations:

– a situation when we know nothing about a statement S, and
– a situation in which we have some arguments in favor of S and equally strong arguments in favor of the opposite statement $\neg S$.

In both situations, we have equally strong arguments in favor of S and in favor of $\neg S$, so it is reasonable to conclude that $d(S) = d(\neg S)$. In the traditional fuzzy logic, when we assume that $d(\neg S) = 1 - d(S)$, this implies that in both situations, we have $d(S) = d(\neg S) = 0.5$. In the intuitionistic fuzzy logic, we describe the situation in which we have no arguments in favor by S by taking $d(S) = 0$, and similarly $d(\neg S) = 0$. Thus, this situation is described differently from the second one when $d(S) = d(\neg S) > 0$.

3 Beyond Intuitionistic Fuzzy

Beyond intuitionistic fuzzy logic: a natural next step. To get an even more adequate description of expert's knowledge, we need to also elicit the expert's degree of confidence in some more complex composite statements.

As we have mentioned, the fewer statements are used in a propositional combination, and the fewer propositional connectives are used, the simpler the combination. If we use one statement S, then the only possible propositional combination is $\neg S$ – which is handled in the intuitionistic fuzzy approach. Thus, if we want to go beyond intuitionistic fuzzy, we need to consider propositional combinations of two original statements S and S'. Among such combinations, the simplest case if when we use a single propositional connective. Thus, the simplest such combinations are combinations of the type $S \,\&\, S'$ and $S \vee S'$.

So, we arrive at the following natural description of the next step: in addition to eliciting, from the experts, their degrees of belief in the original statements S_i, we also elicit their degrees of belief in composite statements $S_i \,\&\, S_j$ and $S_i \vee S_j$. Since we have already included negation, it thus makes sense to also consider the expert's degrees of belief combinations of the type $\neg S_i \,\&\, S_j$, $\neg S_i \,\&\, \neg S_j$, $\neg S_i \vee S_j$, and $\neg S_i \vee \neg S_j$.

The idea in more detail. To describe an imprecise ("fuzzy") property P (e.g., "small"), in the traditional fuzzy logic, to each possible value x of the corresponding quantity, we assign the degree $\mu_P(x) \stackrel{\text{def}}{=} d(P(x))$ to which this quantity satisfies the property P. The corresponding function $\mu_P(x)$ from real values to the interval $[0,1]$ is known as the *membership function*, or, alternatively, as the *fuzzy set*.

In the intuitionistic fuzzy logic, to describe a property P, we need to assign, for each x, two degrees:

– the degree $d(P(x)) \in [0,1]$ that the quantity x satisfies the property P, and
– the degree $d(\neg P(x)) \in [0,1]$ that the quantity x does not satisfy the property P.

This pair of functions forms an *intuitionistic fuzzy set*.

In the new approach, to describe an imprecise property P, we need to also assign, to every pair of values x and x':

– the degree $d(P(x) \,\&\, P(x')) \in [0,1]$ that both quantities x and x' satisfy the property P;
– the degree $d(P(x) \vee P(x')) \in [0,1]$ that either the quantity x or the quantity x' satisfies the property P;
– the degree $d(\neg P(x) \,\&\, P(x')) \in [0,1]$ that the quantity x does not satisfy the property P while the quantity x' satisfies P;
– the degree $d(\neg P(x) \,\&\, \neg P(x')) \in [0,1]$ that neither x nor x' satisfy the property P;
– the degree $d(\neg P(x) \vee P(x')) \in [0,1]$ that either x does not satisfy P or x' satisfies P; and
– the degree $d(\neg P(x) \vee \neg P(x')) \in [0,1]$ that either x or x' does not satisfy the property P.

The resulting collection of functions form the corresponding generalization of the notion of a fuzzy set.

An interesting difference emerges when we want to consider two possible properties P and P'. In both traditional fuzzy approach and intuitionistic fuzzy approach, all we can do is describe these two properties one by one. In the new approach, we also need to describe the relation between the two properties. For example, for each x and x', we can now describe the degree $d(P(x) \,\&\, P(x'))$ to which x satisfies the property P and x' satisfies the property P'.

Comment. The idea of describing such degrees was first formulated – in the probabilistic context – in [5]; see also [9].

This ideas also makes perfect sense. The above idea enables us to describe features of the properties like "small" which are difficult to describe otherwise. For example, while different experts may disagree on which values are small and which are not small, all the experts agree that if x is small and x' is smaller than x, then x' is small as well. In other words, if $x' < x$, then it is not reasonable to believe that x is small but the smaller value x' is not small. In other words, for $P =$"small" and $x' < x$, the corresponding degree of belief $d(P(x) \,\&\, \neg P(x'))$ should be equal to 0.

This possibility is in contrast to the traditional fuzzy logic, where from $d(P(x)) > 0$ and $d(\neg P(x')) = 1 - P(x') > 0$, we would conclude that $d(P(x) \,\&\, \neg P(x')) \approx f_\&(d(P(x)), d(\neg P(x')))$. For most frequently used t-norms such as $f_\&(a, b) = \min(a, b)$ and $f_\&(a, b) = a \cdot b$, from $d(P(x)) > 0$ and $d(\neg P(x')) > 0$, we deduce that the resulting estimate for $d(P(x) \,\&\, \neg P(x'))$ is also positive – and not equal to 0 as common sense tells us it should.

We can go further. To get an even more adequate representation of expert knowledge, we can also elicit expert;s degrees of belief in composite statements which combine three or more original statements S_i.

4 From Type-1 to Type-2 Fuzzy

Need for type-2: brief reminder. We are interested in situations in which an expert is not 100% certain about, e.g., the value of the corresponding quantity. In this case, we use, e.g., estimation on a scale to gauge the expert's degree of belief in different statements. The traditional fuzzy approach assumes that an expert can describe his/her degree of belief by a single number.

In reality, of course, the expert is uncertain about his/her degree of certainty – just like the same expert is uncertain about the actual quantity. In this case, the expert's degree of certainty $d(P(x))$ is no longer a single number – it is, in general, a fuzzy set. This construction, in which, to each x, we assign a fuzzy number $d(P(x))$ is known as a *type-2 fuzzy set*; see, e.g., [7, 8].

Need to combine intuitionistic and type-2 fuzzy sets. It is known that, in many practical situations, the use of type-2 fuzzy sets leads to a more adequate description of expert knowledge. Therefore, to achieve even more adequacy, it is desirable to combine the advantages of type-2 and intuitionistic fuzzy set.

At first glance, such a combination is straightforward. At first glance, it looks like the above combination is straightforward: all the above arguments did not depend on the degree $d(S_i)$ being numbers; the exact same ideas – including the possibility to go beyond the intuitionistic fuzzy sets – can be repeated for the case when the values $d(S_i)$ are themselves fuzzy numbers – or, alternatively, intuitionistic fuzzy numbers.

However, as we will see, the relation between intuitionistic and type-2 fuzzy number is more complicated.

Observation: some intuitionistic fuzzy numbers can be naturally viewed as a particular case of type-2 fuzzy numbers. To explain this unexpected relation, let us start with the simplest possible extension of the classical two-valued logic, in which each statement is either true or false. The more possible truth values we add to the original two, the more complex the resulting logic. Thus, the simplest possible non-classical logic is obtained if we add, to the two classical truth values "true" and "false", the smallest possible number of additional truth values – one. A natural interpretation of this new truth value is "uncertain". For simplicity, let us denote the corresponding truth values by T ("true"), F ("false"), and U ("uncertain").

To fully describe the resulting 3-valued logic, we need to supplement the known truth tables for logical operations involving T and F with operations including the "uncertain" degree U.

For negation, this means adding $\neg U$. For each truth value X, the meaning of $\neg X$ is straightforward: if our degree of belief $d(S)$ in a statement S is equal to X, then our degree of belief in its negation $\neg S$ should be equal to $\neg X$. For "uncertain", the truth value $d(S) = U$ means that we are not sure whether the statement S is true or false. In this case, we are equally uncertain about whether the negation $\neg S$ is true or false; thus, $d(\neg S) = U$. In other words, we have $\neg U = U$.

Similarly, if we are uncertain about S, but we know that S' is false, then the conjunction $S \,\&\, S'$ is also false; thus, $U \,\&\, F = F$. On the other hand, if we know that S' is true (or if we are uncertain about S'), then, depending on whether S is actually true or false, it is possible that the conjunction $S \,\&\, S'$ is true and it is also possible that this conjunction is false. Thus, we have $U \,\&\, T = U \,\&\, U = U$.

If we are uncertain about S, but we know that S' is true, then the disjunction $S \vee S'$ is also true; thus, $U \vee T = T$. On the other hand, if we know that S' is false (or if we are uncertain about S'), then, depending on whether S is actually true or false, it is possible that the disjunction $S \vee S'$ is true and it is also possible that this disjunction is false. Thus, we have $U \vee F = U \vee U = U$.

In the spirit of type-2 logic, instead of selecting one of the three truth values T, F, or U, we can assign *degrees of certainty* $d(T) \geq 0$, $d(F) \geq 0$, and $d(U) \geq 0$ to these three values. One possible way to assign such degrees is to distribute the same fixed amount of degree (e.g., 1) between these three options; in this case, we always have $d(T) + d(F) + d(U) = 1$. Because of this relation, the triple $(d(T), d(F), d(U))$ can be uniquely described by two values $d(T) \geq 0$ and $d(F) \geq 0$ for which $d(T) + d(F) \leq 1$; one can easily see that this is exactly the definition of an intuitionistic fuzzy degree [1, 2].

Moreover, we will show that even some operations on intuitionostic fuzzy degrees can be thus interpreted. Indeed, if we know the triples $(d(T), d(F), d(U))$ and $(d'(T), d'(F), d'(U))$ describing the expert's degree of belief in statements S and S', then the triple $(d''(T), d''(F), d''(U))$ corresponding to the composite statements $S'' = \neg S$, $S'' = S \,\&\, S'$, and $S'' = S \vee S'$ can be obtained by using Zadeh's extension principle. Let us describe this in detail.

In the 3-valued logic, $S'' = \neg S$ is true if and only if S is false, and $S'' = \neg S$ is false if and only if S is true. Thus, $d''(T) = d(F)$ and $d''(F) = d(T)$). This is in line with the usual definition of negation in the intuitionistic fuzzy logic, as $f_\neg((d(T), d(F))) = (d(F), d(T))$.

In the 3-valued logic, $S'' = S \,\&\, S'$ is true if and only if S is true and S' is true:

$$S'' \text{ is } T \Leftrightarrow ((S \text{ is } T) \,\&\, (S' \text{ is } T)).$$

We know the degree $d(T)$ to which S is true, and we know the degree $d'(T)$ to which S' is true. Thus, by applying an appropriate "and"-operation (t-norm), we can conclude estimate the desired degree $d''(T)$ that S'' is true as $f_\&(d(T), d(T'))$. In particular, for a frequently used "and"-operation $f_\&(a, b) = a \cdot b$, we get $d''(T) = d(T) \cdot d'(T)$.

Similarly, $S'' = S \,\&\, S'$ is false if and only if:

- either S is false and S' can take any possible value,
- or S' is false and S can take any possible value.

Thus:

$$S'' \text{ is } F \Leftrightarrow (((S \text{ is } F) \,\&\, (S' \text{ is } T)) \vee ((S \text{ is } F) \,\&\, (S' \text{ is } U)) \vee$$

$$((S \text{ is } F) \,\&\, (S' \text{ is } F)) \vee ((S \text{ is } T) \,\&\, (S' \text{ is } F)) \vee ((S \text{ is } U) \,\&\, (S' \text{ is } F))).$$

By using the same "and"-operation and a frequently used "or"-operation $f_\vee(a, b) = \min(a + b, 1)$, we get the estimate

$$d''(F) = \min(d(F) \cdot d'(T) + d(F) \cdot d'(U) + d(F) \cdot d'(F) + d(T) \cdot d'(F) + d(U) \cdot d'(U), 1).$$

Substituting $d(U) = 1 - d(T) - d(F)$ into this formula, we conclude that $d''(F) = d(F) + d'(F) - d(F) \cdot d'(F)$. This is in line with the usual definition of an "and"-operation in the intuitionistic fuzzy case as

$$f_\&((d(T), d(F)), (d'(T), d'(F))) = (f_\&(d(T), d'(T)), f_\vee(d(F), d'(F))),$$

where $f_\vee(a, b) \stackrel{\text{def}}{=} 1 - f_\&(1 - a, 1 - b))$. For $f_\&(a, b) = a \cdot b$, we thus get $f_\vee(a, b) = a + b - a \cdot b$, and therefore, $d''(T) = f_\&(d(T), d'(T)) = d(T) \cdot d(T'))$ and $d''(F) = d(F) + d'(F) - d(F) \cdot d'(F)$, exactly as in the above type-2 formulas.

For $S'' = S \vee S'$, we similarly get

$$S'' \text{ is } F \Leftrightarrow ((S \text{ is } F) \,\&\, (S' \text{ is } F)),$$

and thus, $d''(F) = d(F) \cdot d'(F)$. Also, we get

$$S'' \text{ is } T \Leftrightarrow (((S \text{ is } T) \,\&\, (S' \text{ is } T)) \vee ((S \text{ is } T) \,\&\, (S' \text{ is } U)) \vee$$

$$((S \text{ is } T) \,\&\, (S' \text{ is } F)) \vee ((S \text{ is } F) \,\&\, (S' \text{ is } T)) \vee ((S \text{ is } U) \,\&\, (S' \text{ is } T))),$$

and hence, the degree $d''(T)$ is equal to

$$\min(d(T) \cdot d'(T) + d(T) \cdot d'(U) + d(T) \cdot d'(F) + d(U) \cdot d'(T) + d(F) \cdot d'(T), 1) =$$

$$d(T) + d'(T) - d(T) \cdot d'(T).$$

This is also in perfect accordance with the intuitionistic fuzzy operation $f_\vee((d(T), d(F)), (d'(T), d'(F))) = (f_\vee(d(T), d'(T)), f_\&(d(F), d'(F)))$.

Acknowledgments. This work was supported in part by the National Science Foundation grants HRD-0734825, HRD-124212, and DUE-0926721.

References

1. K. Atanassov, "Intuitionistic fuzzy sets", *Fuzzy Sets and Systems*, 1986, Vol. 20, pp. 87–96.
2. K. Atanassov, *Intuitionistic Fuzzy Sets*, Springer-Verlag, Heidelberg, 1999.
3. B. G. Buchanan and E. H. Shortliffe, *Rule Based Expert Systems: The MYCIN Experiments of the Stanford Heuristic Programming Project*, Addison-Wesley, Reading, Massachusetts, 1984.
4. B. C. Cuong and V. Kreinovich, "Picture Fuzzy Sets - a new concept for computational intelligence problems", *Proceedings of the Third World Congress on Information and Communication Technologies WICT'2013*, Hanoi, Vietnam, December 15–18, 2013, pp. 1–6.
5. I. R. Goodman, "Fuzzy sets as equivalence classes of random sets", In: R. Yager et al. (eds.), *Fuzzy Sets and Possibility Theory*, Pergamon Press, Oxford, UK, 1982, pp. 327–432.
6. G. Klir and B. Yuan, *Fuzzy Sets and Fuzzy Logic*, Prentice Hall, Upper Saddle River, New Jersey, 1995.
7. J. M. Mendel, *Uncertain Rule-Based Fuzzy Logic Systems: Introduction and New Directions*, Prentice-Hall, Upper Saddle River, New Jersey, 2001.
8. J. M. Mendel and D. Wu, *Perceptual Computing: Aiding People in Making Subjective Judgments*, IEEE Press and Wiley, New York, 2010.
9. H. T. Nguyen and V. Kreinovich, "How to Fully Represent Expert Information about Imprecise Properties in a Computer System – Random Sets, Fuzzy Sets, and Beyond: An Overview", *International Journal of General Systems*, 2014, Vol. 43, to appear.
10. H. T. Nguyen and E. A. Walker, *A First Course in Fuzzy Logic*, Chapman and Hall/CRC, Boca Raton, Florida, 2006.
11. L. A. Zadeh, "Fuzzy sets", *Information and Control*, 1965, Vol. 8, pp. 338–353.