Simpler-to-Describe Cases are Often More Difficult to Prove: A Possible Explanation

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Abstract

In many areas of mathematics, simpler-to-describe cases are often more difficult to prove. In this paper, we provide examples of such phenomena (Bieberbach's Conjecture, Poincaré Conjecture, Fermat's Last Theorem), and we provide a possible explanation for this empirical fact.

1 Empirical Fact

Simpler-to-describe cases are often more difficult to prove: an empirical fact. In [10], L. Kazdan attracts the reader's attention to the fact that in mathematics, simpler-to-describe cases are often more difficult to prove. He illustrates this phenomenon on the example of the Bieberbach Conjecture, that each analytical function $f(z) = z + a_2 \cdot z^2 + a_3 \cdot z^3 + \ldots$ which is defined on the unit disk $\{z : |z| \le 1\}$ and which is injective on this disk (i.e., $f(z_1) \ne f(z_2)$ whenever $z_1 \ne z_2$) satisfies the inequalities $|a_n| \le n$ for all n. This conjecture also states that for each n, the equality $|a_n| = n$ is attained only for the function

$$f(z) = \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + \ldots$$

For $n = 2$, this statement was proven by L. Bieberbach himself [2].

The next proven case was $n = 3$, for this case the hypothesis was proven in 1923 by C. Löwner [13]. The case $n = 4$ was proven in 1955 by R. Garabedian and M. Schiffer [7]. The original $n = 4$ proof involved tedious computations, but soon, this proof has been simplified [3]. Interestingly, the simplifying idea cannot be applied to $n = 3$. As a result, the $n = 4$ proof is much shorter and simpler than the proof for $n = 3$. In other words, it looks like the simpler-to-describe case $n = 3$ is more difficult to prove.

Such an irregularity continued: the case $n = 6$ turned out to be simpler to prove than the case $n = 5$: the case $n = 6$ was proven in 1968–69 [14, 15], while the case $n = 5$ was proven only in 1972 [16].

The whole problem was solved when the Bieberbach Cojecture was proven by L. de Branges in 1985 [4]; see also [1, 5, 8, 9, 11, 12, 21]. However, a similar phenomenon can be found in many other areas of mathematics:

- In Topology, the Poincaré Conjecture [19] – that every close n-dimensional manifold which has the homotopy type of the n-dimensional sphere is homeomorphic to this sphere – was first proven, by S. Smale, for all dimensions higher than 4 [20]; for $n = 4$, it was proven by M. H. Freedman [6], and the case $n = 3$ was only recently famously solved by G. Perelman [17, 18].

- in Number Theory, the famous Fermat's Last Theorem – that the equation $a^n + b^n = c^n$ has not solutions in natural numbers when $n > 2$ – was first proven by P. Fermat himself for $n = 4$ in 1637, and the case $n = 3$ was proven (by L. Euler) only in 1770. Such "jumps" between values n continued until the Fermat's Last Theorem was finally proven by A. Wiles for all n, in 1995 [23].

How can we explain that simpler-to-describe cases are often more difficult to prove?

2 Possible Explanation

What does it mean that a proof is more difficult? In a reasonable first approximation, a natural way to describe the difficulty of a proof is by its length – as described in some appropriate formal proof system. This is, in effect, what L. Kazdan does in [10].

Let $\ell(S)$ denote the length of the shortest possible proof of a statement S or its negation $\neg S$ in the selected formal system. If a statement S is independent on the selected formal system, so that neither S not its negation $\neg S$ can be proven in this system, then we take $\ell(S) = \infty$.

Precise formulation of the problem. Let us consider a general situation, when we have a sequence of statements $A(1)$, $A(2)$, ..., that we are interested in proving (or disproving). We are interested in the sequence of values $\ell(A(n))$ corresponding to different cases n.

Preliminary analysis of the problem. The very formulation $A(n)$ – and thus, a proof of $A(n)$ – includes the description of the number n itself. Thus, the length of a proof of $A(n)$ cannot be shorter than the length $\approx \log_2(n)$ of describing the number n. Therefore, $\ell(A(n)) \geq \log_2(n)$ and hence, $\ell(A(n)) \to \infty$ when $n \to \infty$.

Crudely speaking, this means that, in general, the length $\ell(A(n))$ increases with n – overall, the complexity of the proof grows. However, this general statement is consistent both:

- with a monotonic growth (which seem to be intuitively expected) and

- with the actually observed growth which, as we mentioned, is often non-monotonic.

How can we explain this non-monotonicity?

A reasonable assumption. In general – just like in the case of the Bieberbach conjecture – there seems to be relation between the statements $A(n)$ and $A(m)$ which correspond to different cases $n \neq m$. We can describe this independence by assuming that if $n \neq m$ and we have proven that $A(n)$ is true, this should not help us prove that $A(m)$ is true or that $A(m)$ is not true – it should not even help us to decide that the statement $A(m)$ is decidable within the given formal theory (i.e., that either the statement $A(m)$ or its negation $\neg A(m)$ can be deduced from this theory).

Let us show that this seemingly reasonable assumption can explain why the sequence $\ell(A(n))$ is often non-monotonic. We will prove this by reduction to a contradiction.

Desired explanation. Let us assume that the length $\ell(A(n))$ of the shortest proof is monotonically increasing with n. In this case, once we have a proof that $A(n)$ is true, then we know that the shortest proof $\ell(A(n))$ cannot exceed the length ℓ_0 of our proof: $\ell(A(n)) \le \ell_0$. Since we assumed that the shortest proof length $\ell(A(x))$ is a monotonic function of x, we thus conclude that for all $m < n$, we have $\ell(A(m)) \le \ell(A(n))$, and therefore, that $\ell(A(m)) \le \ell_0$.

Since the case when $A(m)$ is independent of the given theory corresponds to $\ell(A(m)) = \infty$, the fact that we have $\ell(A(m)) \le \ell_0 < \infty$ means that either the statement $A(m)$ or its negation $\neg A(m)$ can be deduced from the formal theory – which contradicts to our assumption that we cannot deduce this from the proof of $A(n)$. This contradiction shows that the independence assumption is indeed incompatible with monotonicity of $\ell(A(n))$ and therefore, under this assumption, the function $\ell(A(n))$ is, in general, not monotonic.

Comment. Once we know that either $A(m)$ or $\neg A(m)$ are provable, we can, in principle, find the corresponding proof by simply trying all possible combinations of symbols – and in the above case, we know that we will succeed once we have tried all combinations of length $\le \ell_0$.

Conclusion. We have shown that the above seemingly reasonable independence assumption indeed implies that, in general, the length $\ell(A(n))$ of the shortest proof is not monotonically depending on n. Thus, we have values $m < n$ for which $\ell(A(m)) > \ell(A(n))$, i.e., we have situations in which simpler-to-describe cases are more difficult to prove.

Acknowledgments. This work was supported in part by the National Science Foundation grants HRD-0734825 and HRD-1242122 (Cyber-ShARE Center of Excellence) and DUE-0926721.

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