From Interval-Valued Probabilities to Interval-Valued Possibilities: Case Studies of Interval Computation under Constraints

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Abstract. In many engineering situations, we need to make decisions under uncertainty. In some cases, we know the probabilities p_i of different situations i; these probabilities should add up to 1. In other cases, we only have expert estimates of the degree of possibility μ_i of different situations; in accordance with the possibility theories, the largest of these degrees should be equal to 1.

In practice, we often only know these degrees p_i and μ_i with uncertainty. Usually, we know the upper bound and the lower bound on each of these values. In other words, instead of the exact value of each degree, we only know the *interval* of its possible values, so we need to process such interval-valued degrees.

Before we start processing, it is important to find out which values from these intervals are actually possible. For example, if only have two alternatives, and the probability of the first one is 0.5, then – even if the original interval for the second probability is wide – the only possible value of the second probability is 0.5.

Once the intervals are narrowed down to possible values, we need to compute the range of possible values of the corresponding characteristics (mean, variance, conditional probabilities and possibilities, etc.). For each such characteristic, first, we need to come up with an algorithm for computing its range.

In many engineering applications, we have a large amount of data to process, and many relevant decisions need to be made in real time. Because of this, it is important to make sure that the algorithms for computing the desired ranges are as fast as possible.

We present expressions for narrowing interval-valued probabilities and possibilities and for computing characteristics such as mean, conditional probabilities, and conditional possibilities. A straightforward computation of these expressions would take time which is quadratic in the number of inputs n. We show that in many cases, linear-time algorithms are possible – and that no algorithm for computing these expressions can be faster than linear-time.

Keywords: interval-valued probabilities; interval-valued possibilities; interval computations; constraints.

1. Introduction and Motivation

We use knowledge to make decisions. In many real-life situations, we need to make decisions. For example, in a computer server, an intrusion detection system must decide whether a given pattern of behavior represents a possible intrusion – and activate defenses against this intrusion. In medicine, we need to decide on the best way to cure a patient. Each decision is based on our knowledge about the situation.

Empirical knowledge vs. expert knowledge. This knowledge comes from two sources.

First, we have records of previous experiences. For example, in the case of intrusion detection, we have records of someone hacking into a databases, as well as records of a normal functioning of a system. In case of medicine, we have numerous records of different patients with different symptoms and different confirmed diagnoses.

We also have knowledge of experts – which can also be used to make a good decision. For example, an expert can say that a certain pattern of behavior – e.g., when several repeated logins are attempted with different passwords – is a strong indication of an intrusion attempt. A skilled medical doctor can decided that for a given patient, a particular type of cough most probably comes from allergy and not from cold.

Let us describe how to represent and process both types of knowledge.

Empirical knowledge is usually represented in terms of probabilities. In most practical situations, we rarely have all the information which is needed to make a decision; the available information is usually incomplete. As a result, based on this partial information, we often cannot make a definite conclusion about what is happening in the system. In the past, there may have been many different situations similar to what we observe now, and their detailed analysis showed that they have been caused by different phenomena. For example, repeated attempts to log in with different passwords do not necessarily indicate an intrusion – they may come from a legitimate absent-minded user who forgot which of his numerous passwords corresponds to which system and is therefore trying them all.

In such cases, based on our prior experience, we cannot definitely tell what kind of phenomenon we encounter, but we can say how frequently phenomena of different type were encountered in similar past situations. For example, we may know that in situations with repeated logins, in 10% of the cases it was absent-minded users, and in 90% of the cases, it was an intrusion attempt. In other words, based on our prior experience, we know the *probabilities* of different phenomena. This is all the information that we can immediately deduce from the past: if out of 10 patients with a certain type of cough 7 had allergy and 3 had cold, then the only information that we have is that for this type of cough, the probability of an allergy is 70% and the probability of cold is 30%.

Thus, a general way to describe empirical knowledge is by describing the corresponding probabilities. (The case when we are absolute sure about the phenomenon can be viewed as a particular case of this probabilistic description, when the correct diagnosis has probability 1 and all other diagnoses have probability 0).

Need to process probabilities. How can we use the empirical probabilities when making a decision? Because of the above uncertainty, for each possible decision a, we do not know the exact consequences, at best, we know the probabilities $p_1(a), \dots, p_n(a)$ of different outcomes. It has been shown (Fishburn, 1988; Luce and Raiffa, 1989; Raiffa, 1997; Nguyen et al., 2012) that consistent preferences under such uncertainty can be described as follows:

- we assign certain numerical values $u_1, \ldots, u_n$ to different possible outcomes, and then
- a decision a is better than a decision a' if it has higher value of expected utility:

$$\sum_{i=1}^{n} p_i(a) \cdot u_i > \sum_{i=1}^{n} p_i(a') \cdot u_i.$$

From this viewpoint, to compare different possible decisions, we need to be able to compute the corresponding *expected values.*

To compute the expected value, we need to know the probabilities $p_i(a)$. For example, if an intrusion detection system activates its defenses, it may lead to positive consequences – if this was indeed an attack – or it may lead to negative consequence, such as an inconvenient denial of service for a legitimate user. In other words, we need to know the probabilities of different phenomena based on the given situation. In some cases, such probabilities can be determined directly from the empirical records. In other cases, we do not have a direct record of such probabilities, so we must deduce them from whatever information we have.

For example, in medical databases, we usually have records of patients with different diseases, so what we have is probabilities $p(\text{cough} \mid \text{allergy})$ that a person has a cough if this person has allergy or that the person has a cough under the condition that this person has cold. What we want is the *opposite* probabilities, e.g., that a person has allergy if this person has a cough. A well-known way to compute such probabilities is to use the Bayes rule; see, e.g., (Sheskin, 2011). The use of this rule requires that we compute *conditional probabilities* $p(A \mid B) = \dfrac{P(A \,\&\, B)}{P(B)}$.

How to represent expert knowledge? An expert is also rarely 100% sure. In some situations, the expert can estimate the probabilities of different phenomena. However, often, an expert can only provide *partial information* about these probabilities.

Case when expert knows relative probabilities. Often, an expert can only estimate *relative probabilities*, i.e., an expert knows the ratios r_{ij} which are equal to the ratios p_i/p_j, but not the actual values of the probabilities p_i. For example, based on his or her experience, a medical doctor knows that allergy occurs twice more frequently than cold, but he may not know the frequency with which these diseases occur in a general population.

How can we represent this information? If we knew the probability $p_{i_0} > 0$ of one of the phenomena i_0, we can then use the known ratios $r_{ii_0} = p_i/p_{i_0}$ to uniquely determine all other probabilities p_i as $p_i = p_{i_0} \cdot r_{ii_0}$. Let us therefore select one of the phenomena i_0, take its "probability" to be equal to some fixed positive value $v > 0$, and then use the formula $\mu_i \stackrel{\text{def}}{=} v \cdot r_{ii_0}$ to estimate the expert's degree of confidence in the i-th statement.

The most emphasis should be on the most probable phenomenon, the one with the largest probability $p_m = \max\limits_{1 \le i \le n} p_i$, so it makes sense to take $i_0 = m$. To simplify computations, let us select the simplest possible positive number v. The simplest possible positive value is 1, so we take $v = 1$

and thus, $\mu_i = r_{im}$. Since the value p_m was the largest, we have $\mu_i = r_{im} = p_i/p_m \le 1$ for all i and $\mu_m = r_{mm} = p_m/p_m = 1$. Thus, here, we have $\max\limits_{1\le i\le n} \mu_i = 1$. The values $\mu_1, \ldots, \mu_n$ which satisfy this constraint are known as *possibilities*; see, e.g. (Dubois, Lang, and Prade, 1998; Dubois and Prade, 1998; Dubois, Moral, and Prade, 1998).

How to determine conditional possibilities. As we have mentioned earlier, one of the important procedures when processing probabilities is estimation of conditional probabilities. If instead of the actual probabilities, we only know relative probabilities, i.e., the possibilities μ_i, how can we then determine the conditional probabilities?

Suppose that for n different phenomena, we know the possibilities $\mu_1, \ldots, \mu_n$. This means that the actual (unknown) probabilities have the form $p_i = c \cdot \mu_i$, for some unknown value $c = p_m$. Suppose now that we learned that in reality, only phenomena from some set $S \subset \{1, \ldots, n\}$. Then, the corresponding conditional probabilities $p(i \mid S)$ take the following form:

- for $i \in S$, we have $p(i \mid S) = C \cdot p_i$, where $C = \dfrac{1}{p(S)} = \dfrac{1}{\sum\limits_{j\in S} p_j}$, and
- for $i \notin S$, we have $p(i \mid S) = 0$.

Substituting $p_i = c \cdot \mu_i$ into this formula, we conclude that:

- for $i \in S$, we have $p(i \mid S) = C \cdot c \cdot \mu_i$, and
- for $i \notin S$, we have $p(i \mid S) = 0$.

We want to describe the corresponding relative possibilities $\mu(i \mid S) = c_1 \cdot p(i \mid S)$, for some constant c_1. Substituting the above formula for $p(i \mid S)$, we get the following formulas:

- for $i \in S$, we have $\mu(i \mid S) = c_2 \cdot \mu_i$, where $c_2 \stackrel{\text{def}}{=} c_1 \cdot C \cdot c$, and
- for $i \notin S$, we get $\mu(i \mid S) = 0$.

The value c_2 can be determined from the requirement that the largest of the values $\mu(i \mid S)$ be equal to 1, so we get (Dubois, Lang, and Prade, 1998; Dubois and Prade, 1998; Dubois, Moral, and Prade, 1998):

$$\mu(i \mid S) = \begin{cases} \dfrac{\mu_i}{\max\limits_{j\in S} \mu_j} & \text{if } i \in S \\ 0 & \text{if } i \notin S \end{cases} \tag{1}$$

Case when an expert only know the order between probabilities. In other cases, when it comes to comparing rare events, an expert only knows the *order* between different probabilities: for example, the doctor knows that allergy is more frequent than cold, but he or she is not sure whether allergy is twice more frequent or three times more frequent.

An expert can still describe this knowledge in terms of numbers: e.g., by marking some values of a scale – e.g., on a scale from 0 to 10. In this case, since the largest of these numerical values has no specific meaning, it also makes sense to make it equal to 0. The difference between this case and the previous case if that since we do not know relative probabilities either, there is no need to re-scale the smaller values here. So, to describe the expert's knowledge in this case, we use non-negative values $\mu_1, \ldots, \mu_n$ for which $\max\limits_{i=1,\ldots,n} \mu_i = 1$.

To distinguish this case from the case of relative probabilities, the corresponding possibilities μ_i are sometimes called *qualitative* – while the possibilities corresponding to relative probabilities are called *quantitative*.

How to determine conditional possibilities: qualitative case. In this case, we still want to raise the largest possibility value to 1. However, in contrast to the quantitative case, where we wanted to preserve the ratio between the possibility values, there is no such reason in the qualitative case. So, to simplify computations, it makes sense to keep all the other values intact. Thus, we arrive at the following definition of conditional possibility (Dubois, Lang, and Prade, 1998; Dubois and Prade, 1998; Dubois, Moral, and Prade, 1998):

$$\mu(i \mid S) = \begin{cases} 1 & \text{if } i \in S \text{ and } \mu_i = \max\limits_{j \in S} \mu_j \\ \mu_i & \text{if } i \in S \text{ and } \mu_i < \max\limits_{j \in S} \mu_j \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Probabilities and possibilities have indeed been successfully applied. Probabilities and possibilities (both quantitative and qualitative) have been successfully applied in intrusion detection – and in several other applications (Benferhat et al., 2012; Benferhat, da Costa Pereira, and Tettamanzi, 2013; Ayachi, Ben Amor, and Benferhat, 2014; Ayachi, Ben Amor, and Benferhat, 2014a).

Probabilities and possibilities are often only known with interval uncertainty. In practice, we usually only know the probabilities p_i and the possibilities μ_i with uncertainty. Often, we only know the bounds on each of these values, i.e., we know the *intervals* $\left[\underline{p}_i, \overline{p}_i\right]$ and $\left[\underline{\mu}_i, \overline{\mu}_i\right]$ of their possible values.

It is important to take interval uncertainty into account. Because of the ubiquity of interval uncertainty, we need to analyze how this uncertainty affects the results of computations involving these values. In other words, we need to be able to compute the ranges of possible values of conditional probabilities and possibilities, etc. This is the problem that we will analyze in this paper.

In many applications, we have a large amount of data to process, and need for real-time decisions. It is thus important to make the range-computing algorithms as fast as possible.

Need to "narrow" intervals: An important auxiliary problem. Before we start processing, it is important to find out which values from the given intervals are actually possible. For example, if $n = 2$ and $p_1 = 0.5$, then – even if $\left[\underline{p}_2, \overline{p}_2\right] = [0, 1]$ – the only possible value of p_2 is 0.5. This is another problem that we will analyze and solve in this paper; this is a problem that with which we will start our analysis.

2. Narrowing Intervals: An Important Auxiliary Problem

Narrowing intervals: a general formulation of the problem. Let us first formulate the problem of narrowing intervals in general terms. In general, we have a set of intervals $[\underline{x}_1, \overline{x}_1], \ldots, [\underline{x}_n, \overline{x}_n]$ and a constraint $g(x_1, \ldots, x_n) = c$.

In this paper, we consider two types of constraints:

– for probabilities, we consider constraints of the type $\sum\limits_{i=1}^{n} p_i = 1$, i.e., $g(x_1, \ldots, x_n) = x_1 + \ldots + x_n$ and $c = 1$;

– for possibilities, we consider constraints of the type $\max(\mu_1, \ldots, \mu_n) = 1$, i.e., $g(x_1, \ldots, x_n) = \max(x_1, \ldots, x_n)$ and $c = 1$.

Sometimes, as we have mentioned, because of the constraints, not all values p_i from the corresponding intervals are actually possible. Our objective is to find the ranges of possible values. In other words, for each i, we want to compute the "narrowed" interval $[p_i^-, p_i^+]$ defined as follows:

$$[p_i^-, p_i^+] \stackrel{\text{def}}{=}$$

$$\{x_i \in [\underline{x}_i, \overline{x}_i] : \exists x_1 \ldots \exists x_{i-1} \exists x_{i+1} \ldots \exists x_n (x_1 \in [\underline{x}_1, \overline{x}_1] \,\&\, \ldots \,\&\, x_n \in [\underline{x}_n, \overline{x}_n] \,\&\, g(x_1, \ldots, x_n) = c)\}.$$

Narrowing intervals: case of probabilities. Let us first consider the case of probabilities. In this case, we have the following result.

Proposition 1. *For probabilities, with constraint $\sum\limits_{i=1}^{n} p_i = 1$, the set of possible tuples $(p_1, \ldots, p_n)$ is non-empty if and only if $\sum\limits_{i=1}^{n} \underline{p}_i \le 1 \le \sum\limits_{i=1}^{n} \overline{p}_i$. If this set is non-empty, then the i-th narrowed interval has the form*

$$[p_i^-, p_i^+], \text{ where } p_i^- \stackrel{\text{def}}{=} \max\left(\underline{p}_i, 1 - \sum_{j \ne i} \overline{p}_j\right) \text{ and } p_i^+ \stackrel{\text{def}}{=} \min\left(\overline{p}_i, 1 - \sum_{j \ne i} \underline{p}_j\right). \quad (3)$$

Proof. If the set of possible tuples is non-empty, then, for each such tuple, by adding up n inequalities $\underline{p}_i \le 1 \le \overline{p}_i$ and taking into account that $\sum\limits_{i=1}^{n} p_i = 1$, we get the desired inequality $\sum\limits_{i=1}^{n} \underline{p}_i \le 1 \le \sum\limits_{i=1}^{n} \overline{p}_i$. Vice versa, if this double inequality is satisfied, then this is set is non-empty: indeed, the sum $\sum\limits_{i=1}^{n} p_i$ attains values ≤ 1 and ≥ 1 on the box $\left[\underline{p}_1, \overline{p}_1\right] \times \ldots \times \left[\underline{p}_n, \overline{p}_n\right]$. Since the sum is a continuous function, it thus attains the intermediate value 1 for some tuple.

Let us now prove that when the intervals are consistent, the narrowed intervals have the desired form. Let us first show that every possible value p_i belongs to the interval (3). Indeed, for each i, we have $\underline{p}_i \le p_i$ and $p_i \le \overline{p}_i$. Since the probabilities p_i add up to 1, we have $p_i = 1 - \sum\limits_{j \ne i} p_j$. Thus, $p_i \ge 1 - \sum\limits_{j \ne i} \overline{p}_j$ and $p_i \le 1 - \sum\limits_{j \ne i} \underline{p}_j$. By combining all these bounds on p_i, we conclude that each possible value p_i belongs to the desired interval (3).

Let us prove that, vice versa, every value p_i from the interval (3) is indeed possible. To prove this, for each $j \ne i$, we will take $p_j = \underline{p}_j + \alpha \cdot (\overline{p}_j - \underline{p}_j)$, for an appropriate value $\alpha \in [0,1]$. Once $0 \le \alpha \le 1$, we thus have $\underline{p}_j \le p_j \le \overline{p}_j$, so to complete the proof, we need to find α from the condition that the sum of all the probabilities add up to 1, i.e., that $p_i + \sum\limits_{j \ne i} p_j = 1$. Substituting our expression for p_j into this formula, we conclude that

$$p_i + \sum_{j \ne i} \underline{p}_j + \alpha \cdot \left(\sum_{j \ne i} \overline{p}_j - \sum_{j \ne i} \underline{p}_j \right) = 1. \tag{4}$$

For $\alpha = 0$, the left-hand side of this equality is equal to $p_i + \sum\limits_{j \ne i} \underline{p}_j$. Since the value p_i is within the interval (3), we have $p_i \le p_i^+ = \min \left(\overline{p}_i, 1 - \sum\limits_{j \ne i} \underline{p}_j \right)$, thus $p_i \le 1 - \sum\limits_{j \ne i} \underline{p}_j$ and $p_i + \sum\limits_{j \ne i} \underline{p}_j \le 1$. Thus, for $\alpha = 0$, the left-hand side of the formula (4) is smaller than or equal to 1.

For $\alpha = 1$, the left-hand side of the formula (4) is equal to $p_i + \sum\limits_{j \ne i} \overline{p}_j$. Since the value p_i is within the interval (3), we have $p_i \ge p_i^- = \max \left(\underline{p}_i, 1 - \sum\limits_{j \ne i} \overline{p}_j \right)$, thus $p_i \ge 1 - \sum\limits_{j \ne i} \overline{p}_j$ and $p_i + \sum\limits_{j \ne i} \overline{p}_j \ge 1$. Thus, for $\alpha = 1$, the left-hand side of the formula (4) is greater than or equal to 1.

Since for $\alpha = 0$ the linear expression in the left-hand side of (4) is ≤ 1 and for $\alpha = 1$, this expression is ≥ 1, there exists a value α for which this expression is equal to 1. For this value, the corresponding probabilities p_j are within the corresponding intervals and add up to 1 (i.e., satisfy the constraint). The proposition is proven.

How to compute the narrowed interval: probabilistic case. Straightforward computation of the formula (3) takes n steps for each of n narrowed intervals. Thus, in this case, it would take $n \cdot O(n) = O(n^2)$ computational steps to compute all n narrowed intervals.

We can speed up these computations if we first compute if we take into account that the formulas (3) can be described in the following equivalent form

$$[p_i^-, p_i^+] = \left[\max \left(\underline{p}_i, 1 - \overline{P} + \overline{p}_i \right), \min \left(\overline{p}_i, 1 - \underline{P} + \underline{p}_i \right) \right],$$

where $\underline{P} \stackrel{\text{def}}{=} \sum\limits_{i=1}^{n} \underline{p}_i$ and $\overline{P} \stackrel{\text{def}}{=} \sum\limits_{i=1}^{n} \overline{p}_i$. By using these formulas, we can come up with the following faster algorithm.

Narrowed intervals in the probabilistic case: asymptotically optimal algorithm.

- First, we compute the sums $\underline{P} = \sum\limits_{i=1}^{n} \underline{p}_i$ and $\overline{P} = \sum\limits_{i=1}^{n} \overline{p}_i$.
- Then, for each i, we compute the narrowed interval

$$\left[\max\left(\underline{p}_i, 1 - \overline{P} + \overline{p}_i\right), \min\left(\overline{p}_i, 1 - \underline{P} + \underline{p}_i\right)\right].$$

Each of the two stages takes $O(n)$ steps, so we compute all narrowed intervals in $O(n)$ steps. This algorithm is asymptotically optimal: we need to compute n intervals, therefore we cannot use fewer than $O(n)$ steps.

Narrowing intervals: case of possibilities. In the case of possibilities, we have n intervals $\left[\underline{\mu}_i, \overline{\mu}_i\right]$. We consider all possible tuples $(\mu_1, \ldots, \mu_n)$ such that $\mu_i \in \left[\underline{\mu}_i, \overline{\mu}_i\right]$ and $\max\limits_i \mu_i = 1$.

Proposition 2. *A sequence of possibility intervals* $\left[\underline{\mu}_i, \overline{\mu}_i\right] \subseteq [0, 1]$ *is consistent if and only if* $\max\limits_i \overline{\mu}_i = 1$. *If this sequence is consistent, then the corresponding narrowed intervals have the following form:*

- *if we have several intervals with* $\overline{\mu}_j = 1$, *then there is no narrowing:* $\left[\mu_i^-, \mu_i^+\right] = \left[\underline{\mu}_i, \overline{\mu}_i\right]$ *for each* i*;*
- *if there is only one interval with* $\overline{\mu}_j = 1$, *then for this interval,* $\left[\mu_j^-, \mu_j^+\right] = [1, 1]$, *while for all other intervals* $i \neq j$, *there is no narrowing:* $\left[\mu_i^-, \mu_i^+\right] = \left[\underline{\mu}_i, \overline{\mu}_i\right]$.

Proof. We must have $\mu_i = 1$ for some i, so, since $\mu_i \le \overline{\mu}_i \le 1$, we must have $\overline{\mu}_i = 1$ for this i. Thus, consistency implies that $\max \overline{\mu}_i = 1$. Vice versa, if $\max\limits_i \overline{\mu}_i = 1$, this means that $\overline{\mu}_i = 1$ for some i. For this i, we can then take $\mu_i = 1$, and any other values from the corresponding intervals for other $j \neq i$. Thus, the condition $\max\limits_i \overline{\mu}_i = 1$ is indeed equivalent to consistency.

If there is only one interval with $\overline{\mu}_j = 1$, this means that we have $\overline{\mu}_i < 1$ and thus, $\mu_i < 1$ for all other i. Since we need to have at least one value of possibility equal to 1, the j-th value should always be equal to 1, so we have $\left[\mu_j^-, \mu_j^+\right] = [1, 1]$.

Let us prove that for every i, each value μ_i from the corresponding intervals $\left[\mu_i^-, \mu_i^+\right]$ are indeed possible. If we have at least two intervals with $\overline{\mu}_j = 1$, then one of them is different from i, so we can take $\mu_j = 1$ for this $j \neq i$, and $\mu_k = \overline{\mu}_k$ for all $k \neq i, j$. One can easily check that these values are within the corresponding intervals $\left[\underline{\mu}_i, \overline{\mu}_i\right]$, and the maximum of these values is equal to 1.

If we have only one j for which $\overline{\mu}_j = 1$, then we take $\mu_j = 1$ for this j, and $\mu_k = \overline{\mu}_k$ for all $k \neq i, j$. The proposition is proven.

3. The Main Problem: Interval Computation under Constraints

Interval computations: reminder. Since we are dealing with interval uncertainty, it is natural to relate to *interval computations* (Jaulin et al., 2001; Moore, Kearfott, and Cloud, 2009) which analyzes computations under this uncertainty. One of the main problems of interval computations is as follows;

- we know an algorithm $f(x_1, \ldots, x_n)$;
- we know the intervals $[\underline{x}_1, \overline{x}_1], \ldots, [\underline{x}_n, \overline{x}_n]$;
- we want to compute the range $\{f(x_1, \ldots, x_n) : x_1 \in [\underline{x}_1, \overline{x}_1] \,\&\, \ldots \,\&\, x_n \in [\underline{x}_n, \overline{x}_n]\}$.

Interval computations under constraints: a general description. In our case, we have additional constraints on values from the corresponding intervals: probabilities must add to one, while the largest of the possibility values should be equal to one. The corresponding problems can therefore be viewed as particular cases of the following general problem of interval computation under constraints:

- we know algorithms $f(x_1, \ldots, x_n)$ and $g(x_1, \ldots, x_n)$, and we know a number c;
- we know the intervals $[\underline{x}_1, \overline{x}_1], \ldots, [\underline{x}_n, \overline{x}_n]$;
- we want to compute the range

$$\{f(x_1, \ldots, x_n) : x_1 \in [\underline{x}_1, \overline{x}_1], \ldots, x_n \in [\underline{x}_n, \overline{x}_n], \text{ and } g(x_1, \ldots, x_n) = c\}.$$

An additional problem: are all combinations possible? Often, we estimate the values of several characteristics $y_1 = f_1(x_1, \ldots, x_n)$, $\ldots$, $y_m = f_m(x_1, \ldots, x_n)$. Interval computation under constraints enable us to find the ranges $\left[\underline{y}_1, \overline{y}_1\right]$, $\ldots$, $\left[\underline{y}_m, \overline{y}_m\right]$ of each of these characteristics.

A natural question is: are all combinations $(y_1, \ldots, y_m)$ of values $y_j \in [\underline{y}_j, \overline{y}_j]$ possible? In other words, can we find $x_i \in [\underline{x}_i, \overline{x}_i]$ for which $g(x_1, \ldots, x_n) = c$ and $y_j = f_j(x_1, \ldots, x_n)$ for all j? If all combinations are possible, then the set S of all combinations $(y_1, \ldots, y_m)$ is equal to the corresponding box:

$$S = B \stackrel{\text{def}}{=} \left[\underline{y}_1, \overline{y}_1\right] \times \ldots \times \left[\underline{y}_m, \overline{y}_m\right].$$

Otherwise, the set S is a proper subset of the box B.

4. Conditional Probabilities $q_i \stackrel{\text{def}}{=} p(i \,|\, S)$

Formulation of the problem.

- we know the intervals $\left[\underline{p}_i, \overline{p}_i\right] \subseteq [0, 1]$; we assume that these intervals have already been narrowed (see above);

– we know the condition S, i.e., a set $S \subset \{1, \ldots, n\}$;

– we want to find, for each $i \in S$, the range $\left[\underline{q}_i, \overline{q}_i\right]$ of possible values of the conditional probability $q_i = \dfrac{p_i}{\sum\limits_{j \in S} p_j}$ when $p_j \in \left[\underline{p}_j, \overline{p}_j\right]$ and $\sum\limits_{j=1}^{n} p_j = 1$.

What if we ignore the constraint. If we ignore the constraint $\sum\limits_{i=1}^{n} p_i = 1$, then the corresponding problem is easy to solve. Indeed, if we divide both the numerator and the denominator of the formula for conditional probability by p_i, we conclude that $q_i = \dfrac{1}{1 + \sum\limits_{j \in S,\ j \neq i} \dfrac{p_j}{p_i}}$. This expression is increasing in p_i and decreasing in all other values p_j, $j \neq i$. Thus, the smallest value of q_i is attained when p_i is the smallest possible and each p_j for $j \neq i$ is the largest possible. Similarly, the largest value of q_i is attained when p_i is the largest possible and each p_j for $j \neq i$ is the smallest possible. So, we get the range $\left[\dfrac{\underline{p}_i}{\underline{p}_i + \sum\limits_{j \in S,\ j \neq i} \overline{p}_j}, \dfrac{\overline{p}_i}{\overline{p}_i + \sum\limits_{j \in S,\ j \neq i} \underline{p}_j}\right]$.

If we take constraints into account, not all such values are possible. Let us take $n = 10$ and $\left[\underline{p}_i, \overline{p}_i\right] = [0, 0.2]$ for all i. One can easily check that these intervals are already narrowed – in the sense that applying the above narrowing operation to these intervals leaves them intact. Let us take $S = \{1, \ldots, 9\}$. Then, the above formula leads to the upper bound $q_1 = \dfrac{\overline{p}_1}{\overline{p}_1 + \sum\limits_{j \neq 1} \underline{p}_j} = \dfrac{0.2}{0.2 + 0 + \ldots + 0} = 1$. However, this value is only attained when $p_1 = 0.2$ and $p_2 = \ldots = p_9 = 0$. In this case, however, $\sum\limits_{i=1}^{10} p_i = 0.2 + 0 + \ldots + 0 + p_{10} \leq 0.4$, while this sum should be equal to 1.

It is therefore desirable to come up with bounds that take constraints into account.

Proposition 3. *The range of possible values of the conditional probability is equal to*

$$[\underline{1}_i, \overline{q}_i] = \left[\frac{\underline{p}_i}{\underline{p}_i + \min\left(\sum\limits_{j \in S,\ j \neq i} \overline{p}_j, 1 - \sum\limits_{k \notin S} \underline{p}_k - \underline{p}_i\right)}, \frac{\overline{p}_i}{\overline{p}_i + \max\left(\sum\limits_{j \in S,\ j \neq i} \underline{p}_j, 1 - \sum\limits_{k \notin S} \overline{p}_k - \overline{p}_i\right)}\right].$$

Proof. We have already shown that

$$q_i = \frac{1}{1 + \dfrac{\sum\limits_{j \in S,\ j \neq i} p_j}{p_i}},$$

i.e., $q_i = \dfrac{1}{1 + \dfrac{1}{r_i}}$, where $r_i \stackrel{\text{def}}{=} \dfrac{p_i}{\sum\limits_{j\in S,\ j\neq i} p_j}$. Thus:

- the value q_i is the smallest if and only if the ratio r_i is the smallest and
- the value q_i is the largest when the ratio r_i is the largest.

From the fact that $p_j \in \left[\underline{p}_j, \overline{p}_j\right]$ for every j, we conclude that $\underline{p}_i \le p_i \le \overline{p}_i$, that

$$\sum_{j\in S,\ j\neq i} \underline{p}_j \le \sum_{j\in S,\ j\neq i} p_j \le \sum_{j\in S,\ j\neq i} \overline{p}_j,$$

and $\sum\limits_{k\notin S} \underline{p}_k \le \sum\limits_{k\notin S} p_j \le \sum\limits_{k\notin S} \overline{p}_k$. Since $p_i + \sum\limits_{j\in S,\ j\neq i} p_j + \sum\limits_{k\notin S} p_k = 1$, thus $\sum\limits_{j\in S,\ j\neq i} p_j = 1 - \sum\limits_{k\notin S} p_k - p_i$, and the inequalities for $\sum\limits_{k\notin S} p_k$ imply that

$$1 - \sum_{k\notin S} \overline{p}_k - p_i \le \sum_{j\in S,\ j\neq i} p_j \le 1 - \sum_{k\notin S} \underline{p}_k - p_i.$$

For each value $p_i \in [\underline{p}_i, \overline{p}_i]$, the ratio r_i is the smallest when the sum $\sum\limits_{j\in S,\ j\neq i} p_j$ is the largest. This sum is bounded from above by two bounds: $\sum\limits_{j\in S,\ j\neq i} \overline{p}_j$ and $1 - \sum\limits_{k\notin S} \underline{p}_k - p_i$. Thus,

$$\sum_{j\in S,\ j\neq i} p_j \le \min\left(\sum_{j\in S,\ j\neq i} \overline{p}_j, 1 - \sum_{k\notin S} \underline{p}_k - p_i\right).$$

So, for a fixed value p_i, the largest possible value of the sum $\sum\limits_{j\in S,\ j\neq i} p_j$ is equal to

$$\sum_{j\in S,\ j\neq i} p_j = \min\left(\sum_{j\in S,\ j\neq i} \overline{p}_j, 1 - \sum_{k\notin S} \underline{p}_k - p_i\right).$$

Thus, the smallest possible value of the ratio r_i is equal ro

$$\frac{p_i}{\min\left(\sum\limits_{j\in S,\ j\neq i} \overline{p}_j, 1 - \sum\limits_{k\notin S} \underline{p}_k - p_i\right)}.$$

As p_i increases, the numerator of this fraction increases and the denominator decreases, so the fraction itself increases. Thus, the smallest possible value $\underline{r}_i$ of the ratio r_i is attained when p_i attains its smallest possible value $\underline{p}_i$. Substituting this smallest value

$$r_i = \frac{\underline{p}_i}{\min\left(\sum\limits_{j\in S,\ j\neq i} \overline{p}_j, 1 - \sum\limits_{k\notin S} \underline{p}_k - \underline{p}_i\right)}$$

into the formula describing q_i in terms of r_i, and multiplying both the numerator and the denominator of the resulting formula by $\underline{p}_i$, we get the desired expression for $\underline{q}_i$.

Similarly, we get the formula for $\overline{q}_i$. The proposition is proven.

Computing conditional probabilities: asymptotically optimal algorithm. Straightforward computations would require quadratic time: linear time for each of n values $\underline{q}_i$ and $\overline{q}_i$. However, we can compute these value in asymptotically optimal linear time if we reformulate the above formulas in the equivalent form

$$\left[\underline{q}_i, \overline{q}_i\right] = \left[\frac{\underline{p}_i}{\underline{p}_i + \min\left(\overline{P}_S - \overline{p}_i, 1 - \underline{P}_{-S} - \underline{p}_i\right)}, \frac{\overline{p}_i}{\overline{p}_i + \max\left(\underline{P}_S - \underline{p}_i, 1 - \overline{P}_{-S} - \overline{p}_i\right)}\right],$$

where we denoted $\overline{P}_S \stackrel{\text{def}}{=} \sum\limits_{j \in S} \overline{p}_j$, $\underline{P}_S \stackrel{\text{def}}{=} \sum\limits_{j \in S} \underline{p}_j$, $\overline{P}_{-S} \stackrel{\text{def}}{=} \sum\limits_{k \notin S} \overline{p}_k$, and $\underline{P}_{-S} \stackrel{\text{def}}{=} \sum\limits_{k \notin S} \underline{p}_k$.

Not all combinations of possible conditional probabilities are possible. Let us consider the case when $n = 8$, $\left[\underline{p}_i, \overline{p}_i\right] = [0.1, 0.15]$ for all i, and $S = \{1, 2, 3, 4\}$. In this case, it is possible to have $q_1 = \dfrac{1}{3} = \dfrac{0.15}{0.15 + 0.1 + 0.1 + 0.1}$: indeed, we can take $p_1 = 0.15$, $p_2 = p_3 = p_4 = 0.1$, $p_5 = p_6 = p_7 = 0.15$, and $p_8 = 0.1$. It is also possible to have $q_1 = \dfrac{2}{11} = \dfrac{0.1}{0.1 + 0.15 + 0.15 + 0.15}$: indeed, we can take $p_1 = 0.1$, $p_2 = p_3 = p_4 = 0.15$, $p_5 = p_6 = p_7 = 0.1$, and $p_8 = 0.15$.

Similarly, we can have $q_1 = \dfrac{1}{3}$ as a possible value of q_1, $q_2 = \dfrac{1}{3}$ as a possible value of q_2, $q_3 = \dfrac{2}{11}$ as a possible value of q_3, and $q_4 = \dfrac{2}{11}$ as a possible value of q_4. However, as we will prove, there are no values p_i for which $q_i = \dfrac{p_i}{p_1 + p_2 + p_3 + p_4}$ for all $i = 1, 2, 3, 4$. Indeed, due to monotonicity, the only way to have $q_1 = \dfrac{1}{3}$ is to have $p_1 = 0.15$ and $p_2 = p_3 = p_4 = 0.1$. However, $q_2 = \dfrac{1}{3}$ is only possible for $p_1 = 0.1$ – a contradiction.

5. Quantitative Conditional Possibility: Interval Case

Formulation of the problem.

- we know n intervals $\left[\underline{\mu}_i, \overline{\mu}_i\right]$; we assume that these intervals have already been narrowed (see above);
- we know the condition S, i.e., a set $S \subset \{1, \ldots, n\}$;
- we want to find, for each $i \in S$, the range $\left[\underline{q}_i, \overline{q}_i\right]$ of possible values of the quantitative conditional possibility $q_i = \dfrac{\mu_i}{\max\limits_{j \in S} \mu_j}$ when $\mu_j \in \left[\underline{\mu}_j, \overline{\mu}_j\right]$ and $\max\limits_{1 \le j \le n} \mu_j = 1$.

Proposition 4.

- *If the set S contains all the indices i for which $\overline{\mu}_i = 1$, then $\underline{q}_i = \underline{\mu}_i$ and $\overline{q}_i = \overline{\mu}_i$ for all $i \in S$.*
- *In all other cases,* $\underline{q}_i = \dfrac{\underline{\mu}_i}{\max\left(\underline{\mu}_i, \max\limits_{j\in S,\ j\neq i} \overline{\mu}_j\right)}$ *and* $\overline{q}_i = \dfrac{\overline{\mu}_i}{\max\left(\overline{\mu}_i, \max\limits_{j\in S,\ j\neq i} \underline{\mu}_j\right)}$.

Proof. Each tuple of possibility values μ_i has to satisfy the constraint $\max\limits_i \mu_i = 1$. So, for each tuple, there must be an index i for which μ_i. Since $\mu_i \le \overline{\mu}_i = 1$, this implies that $\overline{\mu}_i = 1$. So, if all indices i with $\overline{\mu}_i = 1$ are contained in the set S, this means that for every tuple of possibility values, we have $\mu_i = 1$ for some $i \in S$. In this case, $\max\limits_{i\in S} \mu_i = 1$ and therefore, formula (1) reduces to $q_i = \mu(i \,|\, S) = \mu_i$. Thus, in this case, the range of possible value of q_i coincides with the range of possible values of μ_i.

Let us now consider the case when there is at least one index $k \notin S$ for which $\overline{\mu}_k = 1$. Whether the inequality $\mu_i < \max\limits_{j\in S,\ j\neq i} \mu_j$ is satisfied or not, the expression for the conditional possibility q_i (non-strictly) increases with μ_i and decreases with μ_j for $j \in S, j \neq i$. Thus, each value q_i is larger than or equal to the value corresponding to the smallest possible value of μ_i and to the largest possible values of μ_j, and similarly, each value q_i is smaller than or equal to the value corresponding to the largest possible value of μ_i and to the smallest possible values of μ_j:

$$\frac{\underline{\mu}_i}{\max\left(\underline{\mu}_i, \max\limits_{j\in S,\ j\neq i} \overline{\mu}_j\right)} \le q_i = \frac{\mu_i}{\max\limits_{j\in S} \mu_j} \le \frac{\overline{\mu}_i}{\max\left(\overline{\mu}_i, \max\limits_{j\in S,\ j\neq i} \underline{\mu}_j\right)}.$$

Hence, each possible value of q_i indeed belongs to the above interval.

To complete the proof, we thus need to show that each value from the above interval can be represented as $q_i = \dfrac{\mu_i}{\max\limits_{j\in S} \mu_j}$ for an appropriate tuple μ_i. The value $\dfrac{\underline{\mu}_i}{\max\left(\underline{\mu}_i, \max\limits_{j\in S,\ j\neq i} \overline{\mu}_j\right)}$ can be obtained if we take $\mu_i = \underline{\mu}_i$ and $\mu_j = \overline{\mu}_j$ for all $j \neq i$. Since this includes the value k for which $\overline{\mu}_k = 1$, we thus get $\mu_k = 1$ and $\max\limits_j \mu_j = 1$.

The value $\dfrac{\overline{\mu}_i}{\max\left(\overline{\mu}_i, \max\limits_{j\in S,\ j\neq i} \underline{\mu}_j\right)}$ can be obtained if we take $\mu_i = \overline{\mu}_i$, $\mu_j = \underline{\mu}_j$ for all $j \in S, j \neq i$, and $\mu_k = \overline{\mu}_k$ for $k \notin S$. Since this includes the value k for which $\overline{\mu}_k = 1$, we thus get $\mu_k = 1$ and $\max\limits_j \mu_j = 1$.

The expression for the conditional possibility p_i is a continuous function of the values μ_j, and its domain $\left\{(\mu_1, \ldots, \mu_n) : \max\limits_i \mu_i = 1\right\}$ is a connected set. Thus, the range $\left[\underline{q}_i, \overline{q}_i\right]$ of this function on this domain is a connected set and hence, with any two points it contains all the points in between. Therefore, all the values q_i from the above interval are indeed possible values of the i-th conditional possibility. The proposition is proven.

How to compute quantitative conditional possibilities under interval uncertainty: analysis of the problem. The above formula provide a straightforward algorithm for computing $\underline{q}_i$ and $\overline{q}_i$. For each index i, this algorithm takes $O(n)$ steps, to compute the corresponding maxima. So, the overall computation time of this algorithm is $n \cdot O(n) = O(n^2)$.

How can we compute these bounds faster? The possibility of faster computations comes from the fact that if we denote by $\overline{M}$ the largest of the values $\overline{\mu}_j$ when $j \in S$, and by $\overline{S}$ the second largest of these values, then:

- if $\overline{\mu}_i = \overline{M}$, we have $\max\limits_{j \in S,\ j \neq i} \overline{\mu}_j = \overline{S}$;
- otherwise, if $\overline{\mu}_i < \overline{M}$, then we have $\max\limits_{j \in S,\ j \neq i} \overline{\mu}_j = \overline{M}$.

Thus, we arrive at the following algorithm.

How to compute quantitative conditional possibilities under interval uncertainty: asymptotically optimal algorithm.

- If the set S contains all indices i for which $\overline{\mu}_i = 1$, then we return the values $\underline{q}_i = \underline{\mu}_i$ and $\overline{q}_i = \overline{\mu}_i$ for all $i \in S$.
- Otherwise, we compute the largest $\overline{M}$ and the second largest $\overline{S}$ of the values $\overline{\mu}_j$ corresponding to $j \in S = \{j_1, \ldots, j_m\}$. For that, for every $k = 2, \ldots, m$, we compute the largest $\overline{M}_k$ and the second largest $\overline{S}_k$ of the values $\overline{\mu}_{i_1}, \ldots, \overline{\mu}_{i_k}$ as follows:
  - we start with $\overline{M}_2 = \max\left(\overline{\mu}_{i_1}, \overline{\mu}_{i_2}\right)$ and $\overline{S}_2 = \min\left(\overline{\mu}_{i_1}, \overline{\mu}_{i_2}\right)$;
  - then, once we know $\overline{M}_{k-1}$ and $\overline{S}_{k-1}$, we compute the next values $\overline{M}_k$ and $\overline{S}_k$ as follows:
    * if $\overline{\mu}_{i_k} \geq \overline{M}_{k-1}$, we take $\overline{M}_k = \overline{\mu}_{i_k}$ and $\overline{S}_k = \overline{M}_{k-1}$;
    * if $\overline{S}_{k-1} < \overline{\mu}_{i_k} < \overline{M}_{k-1}$, we take $\overline{M}_k = \overline{M}_{k-1}$ and $\overline{S}_k = \overline{\mu}_{i_k}$;
    * finally, if $\overline{\mu}_{i_k} \leq \overline{S}_{k-1}$, then the values do not change: $\overline{M}_k = \overline{M}_{k-1}$ and $\overline{S}_k = \overline{S}_{k-1}$.

  We then take $\overline{M} = \overline{M}_n$ and $\overline{S} = \overline{S}_n$.
- We also compute the largest $\underline{M}$ and the second largest $\underline{S}$ of the values $\underline{\mu}_j$ corresponding to $j \in S = \{j_1, \ldots, j_m\}$. For that, for every $k = 2, \ldots, m$, we compute the largest $\underline{M}_k$ and the second largest $\underline{S}_k$ of the values $\underline{\mu}_{i_1}, \ldots, \underline{\mu}_{i_k}$ as follows:
  - we start with $\underline{M}_2 = \max\left(\underline{\mu}_{i_1}, \underline{\mu}_{i_2}\right)$ and $\underline{S}_2 = \min\left(\underline{\mu}_{i_1}, \underline{\mu}_{i_2}\right)$;
  - then, once we know $\underline{M}_{k-1}$ and $\underline{S}_{k-1}$, we compute the next values $\underline{M}_k$ and $\underline{S}_k$ as follows:
    * if $\underline{\mu}_{i_k} \geq \underline{M}_{k-1}$, we take $\underline{M}_k = \underline{\mu}_{i_k}$ and $\underline{S}_k = \underline{M}_{k-1}$;
    * if $\underline{S}_{k-1} < \underline{\mu}_{i_k} < \underline{M}_{k-1}$, we take $\underline{M}_k = \underline{M}_{k-1}$ and $\underline{S}_k = \underline{\mu}_{i_k}$;
    * finally, if $\underline{\mu}_{i_k} \leq \underline{S}_{k-1}$, then the values do not change: $\underline{M}_k = \underline{M}_{k-1}$ and $\underline{S}_k = \underline{S}_{k-1}$.

We then take $\underline{M} = \underline{M}_n$ and $\underline{S} = \underline{S}_n$.

– For each $i \in S$, we compute $\underline{q}_i$ as follows:

- if $\overline{\mu}_i = \overline{M}$, then we take $\underline{q}_i = \dfrac{\underline{\mu}_i}{\max\left(\underline{\mu}_i, \overline{S}\right)}$;
- if $\overline{\mu}_i < \overline{M}$, then we take $\underline{q}_i = \dfrac{\underline{\mu}_i}{\max\left(\underline{\mu}_i, \overline{M}\right)}$.

– After that, for each $i \in S$, we compute $\overline{q}_i$ as follows:

- if $\underline{\mu}_i = \underline{M}$, then we take $\overline{q}_i = \dfrac{\overline{\mu}_i}{\max\left(\overline{\mu}_i, \underline{S}\right)}$;
- if $\underline{\mu}_i < \underline{M}$, then we take $\overline{q}_i = \dfrac{\overline{\mu}_i}{\max\left(\overline{\mu}_i, \underline{M}\right)}$.

This algorithm is linear time $O(n)$ and is, thus, asymptotically optimal – since we need to handle at least n input intervals.

Not all combinations of possible values $\mu_s(x)$ with $\max\limits_x \mu_s(x) = 1$ are possible: example. Let us take $n = 4$, $S = \{1, 2, 3\}$, $\left[\underline{\mu}_1, \overline{\mu}_1\right] = [0.1, 0.2]$, $\left[\underline{\mu}_2, \overline{\mu}_2\right] = [0.1, 0.5]$, $\left[\underline{\mu}_3, \overline{\mu}_3\right] = [0.1, 0.5]$, and $\left[\underline{\mu}_4, \overline{\mu}_4\right] = [1, 1]$. Then, the above formulas lead to $\left[\underline{q}_1, \overline{q}_1\right] = \left[\dfrac{0.1}{0.5}, 1\right] = [0.2, 1]$, $\left[\underline{q}_2, \overline{q}_2\right] = [0.2, 1]$, and $\left[\underline{q}_3, \overline{q}_3\right] = [0.2, 1]$.

Let us prove that some combinations of the possible values $q_i \in \left[\underline{q}_1, \overline{q}_1\right]$ are not possible. We will prove this for $q_1 = 0.5$, $q_2 = 0.2$, and $q_3 = 1.0$. We need to prove that it is possible to have the values $\mu_i \in \left[\underline{\mu}_i, \overline{\mu}_i\right]$ for which $q_i = \dfrac{\mu_i}{\max\limits_{j=1,2,3} \mu_j}$. Indeed, since $\mu_2 \geq 0.1$ and $\max\left(\mu_1, \mu_2, \mu_3\right) \leq 0.5$, we have $q_2 = \dfrac{\mu_2}{\max\left(\mu_1, \mu_2, \mu_3\right)} \leq \dfrac{0.1}{0.5} = 0.2$, and the only possibility to have $q_2 = 0.2$ is when $\mu_2 = 0.1$ and $\max\left(\mu_1, \mu_2, \mu_3\right) = 0.5$. In this case, since $\mu_1 \leq 0.2$, we have $q_1 = \dfrac{\mu_1}{\max\left(\mu_1, \mu_2, \mu_3\right)} \leq \dfrac{0.2}{0.5} = 0.4$, which contradicts to the fact that $\mu_1 = 0.5$. The statement is proven.

6. Qualitative Conditional Possibility: Interval Case

Formulation of the problem.

– we know n intervals $\left[\underline{\mu}_i, \overline{\mu}_i\right]$; we assume that these intervals have already been narrowed (see above);

- we know the condition S, i.e., a set $S \subset \{1, \ldots, n\}$;
- we want to find, for each $i \in S$, the smallest $\underline{q}_i$ and the largest $\overline{q}_i$ of possible values of the qualitative conditional possibility (2) when $\mu_j \in \left[\underline{\mu}_j, \overline{\mu}_j\right]$ and $\max\limits_{1 \le j \le n} \mu_j = 1$.

Proposition 5.

- *If the set S contains all the intervals for which $\overline{\mu}_i = 1$, then $\underline{q}_j = \underline{\mu}_j$ and $\overline{q}_j = \overline{\mu}_j$ for all j.*
- *Otherwise, if there is at least one interval $k \notin S$ with $\overline{\mu}_i = 1$, then for every $i \in S$, we have the following:*
  - *If $\underline{\mu}_i \ge \max\limits_{j \in S,\ j \ne i} \overline{\mu}_j$, then $\left[\underline{q}_i, \overline{q}_i\right] = [1, 1]$.*
  - *Else, if $\overline{\mu}_i \ge \max\limits_{j \in S} \underline{\mu}_j$, then $\underline{q}_i = \underline{\mu}_i$ and $\overline{q}_i = 1$.*
  - *Otherwise, $\underline{q}_i = \underline{\mu}_i$ and $\overline{q}_i = \overline{\mu}_i$.*

Proof. In the possibility tuple, one of the values μ_i must be equal to 1. For this value i, we have $\overline{\mu}_i = 1$. Thus, if the set S contains all all indices i for which $\overline{\mu}_i = 1$, the value $\mu_i = 1$ will always occur for some $i \in S$. In this case, formula (2) leads to $q_i = \mu_i$, and thus, the smallest and largest values of q_i coincide with the smallest and largest values of μ_i.

If we have $\overline{\mu}_k = 1$ for some $k \notin S$, then we can always satisfy the condition $\max\limits_j \mu_j = 1$ by taking $\mu_k = 1$ and not affecting the values μ_i for $i \in S$. Let us consider this case.

According to the definition (2) of qualitative conditional possibility q_i, the qualitative conditional possibility value q_i is equal either to the original possibility value μ_i or to 1, and the only case when this value is equal to 1 is when μ_i is the largest of all values μ_j corresponding to $j \in S$.

Let us first consider the case when $\underline{\mu}_i \ge \max\limits_{j \in S,\ j \ne i} \overline{\mu}_j$. In this case, we have $\underline{\mu}_i \ge \overline{\mu}_j$ for all $j \ne i$. Thus, $\mu_i \ge \underline{\mu}_i \ge \overline{\mu}_j \ge \mu_j$, so for all possible tuples, we have $\mu_i \ge \mu_j$ for all $j \ne i$. Thus, the i-th possibility value is always the largest and hence, we always get $q_i = 1$.

Let us now consider the case when $\underline{\mu}_i < \max\limits_{j \in S,\ j \ne i} \overline{\mu}_j$ and $\overline{\mu}_i \ge \max\limits_{j \in S} \underline{\mu}_j$ and let us show that in this case, both values $q_i = \underline{\mu}_i$ and $q_i = 1$ are possible.

- To get the value $q_i = \underline{\mu}_i$, we take $\mu_i = \underline{\mu}_i$ and $\mu_j = \overline{\mu}_j$ for all other $j \in S$. Since $\underline{\mu}_i < \max\limits_{j \in S,\ j \ne i} \overline{\mu}_j$, the i-th possibility value is not the largest and therefore, $q_i = \mu_i = \underline{\mu}_i$.
- To get the value $q_i = 1$, we take $\mu_i = \overline{\mu}_i$ and $\mu_j = \underline{\mu}_j$ for all remaining $j \in S$. Since $\overline{\mu}_i \ge \max\limits_{j \in S} \underline{\mu}_j$, the i-th possibility value is the largest and therefore, $q_i = 1$.

Finally, let us consider the remaining case when $\underline{\mu}_i < \max\limits_{j \in S,\ j \ne i} \overline{\mu}_j$ and $\overline{\mu}_i < \max\limits_{j \in S} \underline{\mu}_j$. In this case, the i-th possibility value cannot be the largest – otherwise from $\mu_i \ge \mu_j$ we would conclude that $\overline{\mu}_i \ge \underline{\mu}_j$ for all $j \ne i$, and thus, that $\overline{\mu}_i \ge \max\limits_{j \in S} \underline{\mu}_j$ – which contradicts to the second of the starting inequalities. Since the i-th value is not the largest, we always have $q_i = \mu_i$. Thus, the largest value of q_i is $\overline{\mu}_i$ and the smallest value of q_i is $\underline{\mu}_i$. The proposition is proven.

How to compute qualitative conditional possibilities under interval uncertainty: analysis of the problem. The above formula provide a straightforward algorithm for computing $\underline{q}_i$ and $\overline{q}_i$. For each index i, this algorithm takes $O(n)$ steps, to compute the corresponding maxima. So, the overall computation time of this algorithm is $n \cdot O(n) = O(n^2)$.

To make this algorithm faster, we can use the same ideas that we use for quantitative conditional possibility.

How to compute qualitative conditional possibilities under interval uncertainty: asymptotically optimal algorithm.

- If the set S contains all indices i for which $\overline{\mu}_i = 1$, then we return the values $\underline{q}_i = \underline{\mu}_i$ and $\overline{q}_i = \overline{\mu}_i$ for all $i \in S$.

- Otherwise, we compute the maximum $\underline{M}$ of all the values $\underline{\mu}_i$, $i \in S$.

- Then, we compute the largest $\overline{M}$ and the second largest $\overline{S}$ of the values $\overline{\mu}_i$ corresponding to $i \in S$; this can be done as in the algorithm for the quantitative case.

- After that, for each $i \in S$, we do the following:

  - if $\overline{\mu}_i = \overline{M}$ and $\underline{\mu}_i \geq \overline{S}$, we return $\underline{q}_i = \overline{q}_i = 1$;
  - if $\overline{\mu}_i < \overline{M}$ and $\underline{\mu}_i \geq \overline{M}$, we return $\underline{q}_i = \overline{q}_i = 1$;
  - otherwise, if $\overline{\mu}_i \geq \underline{M}$, we return $\underline{q}_i = \underline{\mu}_i$ and $\overline{q}_i = 1$;
  - for all other $i \in S$, we return $\underline{q}_i = \underline{\mu}_i$ and $\overline{q}_i = \overline{\mu}_i$.

This algorithm takes linear time $O(n)$, and is, therefore, asymptotically optimal – since we need to least $c \cdot n$ computational steps to process all n input intervals.

Not all intermediate values are possible. In contrast to the quantitative case when all values q_i between $\underline{q}_i$ and $\overline{q}_i$ are possible, here many intermediate values are not possible. For example, when $n = 3$, $S = \{1, 2\}$, $\left[\underline{\mu}_1, \overline{\mu}_1\right] = \left[\underline{\mu}_1, \overline{\mu}_1\right] = [0, 0.5]$ and $\left[\underline{\mu}_3, \overline{\mu}_3\right] = [1, 1]$, then we have $\underline{q}_1 = 0$ and $\overline{q}_1 = 1$, but it is not possible to have $q_i = 0.6 \in [0, 1]$, since each value q_i coincides either with 1, or with one of the original values μ_i.

7. Mean under Interval Uncertainty

As we have mentioned, to make a decision, we need to be able to estimate the expected value under interval uncertainty. In precise terms, we want to find the range $\left[\underline{E}, \overline{E}\right]$ of the mean $E =$

$\sum\limits_{i=1}^{n} p_i \cdot x_i$ when $p_i \in \left[\underline{p}_i, \overline{p}_i\right]$ and $\sum\limits_{i=1}^{n} p_i = 1$. The maximum is attained when larger values have larger probability. So, if we sort x_i in increasing order $x_1 \le x_2 \le \ldots \le x_n$, we get

$$\overline{E} = \sum_{i=1}^{k-1} \underline{p}_i \cdot x_i + p_k \cdot x_k + \sum_{i=k+1}^{n} \overline{p}_i \cdot x_i.$$

Here, $p_k = 1 - \sum\limits_{i=1}^{k-1} \underline{p}_i - \sum\limits_{i=k+1}^{n} \overline{p}_i$, so $\underline{p}_k \le p_k \le \overline{p}_k$ implies that $\sum\limits_{i=1}^{k} \underline{p}_i + \sum\limits_{i=k+1}^{n} \overline{p}_i \le 1 \le \sum\limits_{i=1}^{k-1} \underline{p}_i + \sum\limits_{i=k}^{n} \overline{p}_i$. This inequality enables us to find k in linear time.

Since sorting requires time $O(n \cdot \log(n))$ (Cormen et al., 2009), we get total time $O(n \cdot \log(n))$.

Comment. A similar algorithm can compute the lower bound $\underline{E}$.

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