

F-Transform Enhancement of the Sampling Theorem and Reconstruction of Noisy Signals

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Abstract

According to a sampling theorem, any band-limited and continuous signal can be uniquely reconstructed from certain of its values. We show that a reconstruction can be obtained from the set of F-transform components and moreover, the sampling theorem follows as a particular case. A special attention is paid to the case where sample values of a signal come with noise. We show that in the presence of noise, a more accurate reconstruction than that based on the sampling theorem can be obtained, if instead of noised sample values the F-transform components of the signal with respect to a generalized fuzzy partition are used.

Keywords: Fuzzy transform, F-transform, fuzzy partition, Nyquist-Shannon-Kotel'nikov theorem, remove of noise

1. Introduction

The F-transform is very useful in many applications such as signal and image compression, time series prediction, image processing, etc.; see, e.g., [3, 8, 9, 12, 13, 14] and references therein. The technique is based on making projections on linear subspaces of functions with an inner product. The F-transform has two phases: direct and inverse. The former produces a sequence of F-transform components, and the latter brings those components back to the space of originals. The inverse F-transform is lossy, so that the problem how to recover an original function from a sequence of F-transform components is of current interest.

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9 Our inspiration came from the knowledge that a sufficiently smooth functions
10 (e.g., analytic in a region or with all derivatives at a certain point, etc.) can be
11 successively reconstructed from available data. Especially if reconstruction of
12 a function can be obtained from the set of its samples, then the corresponding
13 results are known as *sampling theorems*.

14 In applications to signal processing, it is usual to connect the sampling theo-
15 rem with the names of Shannon, Nyquist and Kotel’nikov (see e.g., [6, 10,
16 11, 17]). However, this result was first discovered by Cauchy in 1841 and then
17 rediscovered by Whittaker (see [1]).

18 In our contribution, we show that a function (we use the word “signal”) that
19 fulfills the same conditions as it the standard sampling theorem (a signal should
20 be continuous and band-limited) can be reconstructed from a countable set of its
21 F-transform components. This result contributes to the theory of F-transforms.

22 In the second part, we analyzed whether the reconstruction from the F-
23 transform components is more advantageous than that from samples. We con-
24 sidered the case where sample values of a signal come with noise. We show
25 that in the presence of noise, a more accurate reconstruction than that based
26 on the sampling theorem can be obtained, if instead of noised sample values
27 the F-transform components of the signal are used. Moreover, we show that a
28 proper choice of a fuzzy partition and a sample step leads to a significant remove
29 of a noise if the latter is reconstructed from the F-transform components. This
30 result contributes to the theory of signals.
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34 2. Nyquist-Shannon-Kotel’nikov Reconstruction

35 *Sample-based signal reconstruction: a practical problem.* The values of physical
36 signals usually change with time t . For each such signals x and certain time
37 moments t , it is desirable to compute the values $x(t)$ from the known (measured)
38 values $x(t_k)$ (samples). The genesis of signal samples is explained below.

39 In practice, for each measuring instrument, there is a time lag h so that
40 after each measurement, the next measurement cannot start earlier than h units
41 (seconds) after. In this case, the best we can do is to measure the signal every h
42 seconds. As a result, we get the values $x(t_k)$ corresponding to moments of time
43 $t_k = t_0 + k \cdot h$, where t_0 is the starting moments, usually $t_0 = 0$. Thus, we get
44 values
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$$46 \dots, x(t_{-1}), x(t_0), x(t_1), x(t_2), \dots \quad (1)$$

47 Based on these measurement results, we want to reconstruct values of the signal
48 x at all moments t from its samples in (1).
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50 *Sample-based reconstruction is not always unique.* It is clear that we cannot
51 uniquely reconstruct an arbitrary signal from certain of its sample values. For
52 example, we can have a periodic signal $x(t) = \sin(\omega \cdot (t - t_0))$ with $\omega = \frac{\pi}{h}$. For
53 this signal, for every integer k , we have
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$$55 x(t_k) = \sin(\omega \cdot (t_k - t_0)) = \sin\left(\frac{\pi}{h} \cdot k \cdot h\right) = \sin(k \cdot \pi) = 0,$$

but the signal $x(t)$ is not identically 0. In this case, based on the sample values $x(t_k) = 0$, it is not possible to distinguish the sinusoidal signal $x(t)$ from another signal $y(t) = 0$ which is identically zero.

The same non-uniqueness is true for sinusoids of higher frequencies $\omega = \frac{\ell \cdot \pi}{h}$ for some integer $\ell > 1$.

Nyquist-Shannon-Kotel'nikov result: for band-limited signals, the reconstruction is unique. A precise formulation of the corresponding result uses the fact that every function x from a certain class of functions (e.g., the class $L_2(\mathbb{R})$ of square integrable functions x for which $\int x^2(t) dt < +\infty$) can be represented by

$$x(t) = \frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} \hat{x}(\omega) \cdot e^{i\omega t} d\omega \quad (2)$$

for an appropriate complex-valued function $\hat{x}(\omega)$, where $i \stackrel{\text{def}}{=} \sqrt{-1}$ and

$$e^{i\alpha} = \cos(\alpha) + i \sin(\alpha).$$

The function $\hat{x}(\omega)$ is the *Fourier transform* of x where

$$\hat{x}(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-i\omega t} dt. \quad (3)$$

Both integrals in (2) and (3) are considered as “limits in the mean”, e.g.,

$$\hat{x}(\omega) = \text{l.i.m.}_{n \rightarrow \infty} \int_{-n}^n x(t) \cdot e^{-i\omega t} dt.$$

To guarantee uniqueness of signal reconstruction from sample values, we have to be sure that the Fourier transform of this signal has no components of angular frequency higher than a certain Ω (the signal is band-limited). The corresponding result is known as Nyquist-Shannon-Kotel'nikov theorem (also as the Sampling Theorem), see [6, 10, 11, 17]. According to [1], Shannon made remarkable applications of sampling theory to communication theory, but the proof of Theorem 1 was discovered by Cauchy in 1841. Below, we reproduce a reduced version of the proof given in [1].

Theorem 1 (Sampling Theorem)

Let $x \in L_2(\mathbb{R})$ be continuous and band-limited, i.e. $\hat{x}(\omega) = 0$ for $|\omega| > \Omega$ where Ω is some constant. Then x can be determined by its values at a discrete set of points:

$$x(t) = \sum_{k=-\infty}^{\infty} x\left(\frac{k\pi}{\Omega}\right) \cdot \frac{\sin(\Omega t - k\pi)}{\Omega t - k\pi}. \quad (4)$$

PROOF: We use the fact that $x \in L_2(\mathbb{R})$ is continuous and band-limited, so that by the inversion formula we have the following equality:

$$x(t) = \frac{1}{2\pi} \int_{-\Omega}^{\Omega} \hat{x}(\omega) e^{i\omega t} d\omega. \quad (5)$$

Because $\hat{x} \in L_2[-\Omega, \Omega]$, it can be expanded in a Fourier series

$$\hat{x}(\omega) = \sum_{k=-\infty}^{\infty} g_k e^{-ik\pi\omega/\Omega} \quad (6)$$

where

$$g_k = \frac{1}{2\Omega} \int_{-\Omega}^{\Omega} \hat{x}(\omega) e^{ik\pi\omega/\Omega} d\omega.$$

By (5),

$$g_k = \frac{\pi}{\Omega} x\left(\frac{k\pi}{\Omega}\right).$$

Substituting (6) into (5), we get (4):

$$\begin{aligned} x(t) &= \frac{1}{2\Omega} \sum_{k=-\infty}^{\infty} x\left(\frac{k\pi}{\Omega}\right) \int_{-\Omega}^{\Omega} e^{i\omega(\Omega t - k\pi)/\Omega} d\omega = \\ &= \frac{1}{2\Omega} \sum_{k=-\infty}^{\infty} x\left(\frac{k\pi}{\Omega}\right) \cdot \Omega \cdot \frac{e^{i\omega(\Omega t - k\pi)/\Omega}}{i(\Omega t - k\pi)} \Big|_{-\Omega}^{\Omega} = \\ &= \sum_{k=-\infty}^{\infty} x\left(\frac{k\pi}{\Omega}\right) \cdot \frac{\sin(\Omega t - k\pi)}{\Omega t - k\pi}, \end{aligned}$$

where we made use of

$$\sin \alpha = \frac{e^{i\alpha} - e^{-i\alpha}}{2i}.$$

□

Corollary 1

Let $x, y \in L_2(\mathbb{R})$ be continuous and band-limited signals, i.e., $\hat{x}(\omega) = \hat{y}(\omega) = 0$ for $|\omega| > \Omega$, where Ω is some constant. Denote $h = \frac{\pi}{\Omega}$ and $t_k = k \cdot h$. If for all $k \in \mathbb{Z}^1$, $x(t_k) = y_k$, then for all t , $x(t) = y(t)$.

In the sequel, we will be using the following notation: $h = \frac{\pi}{\Omega}$, $t_k = \frac{k\pi}{\Omega} = k \cdot h$ and the corresponding to it reconstruction formula:

$$x(t) = \sum_{k=-\infty}^{\infty} x(t_k) \cdot \text{sinc}\left(\frac{t}{h} - k\right), \quad (7)$$

Expression (7) is easily derived from (4) with the help of the real function sinc, such that

$$\text{sinc}(t) \stackrel{\text{def}}{=} \frac{\sin(\pi t)}{\pi t}. \quad (8)$$

¹ \mathbb{Z} denotes the set of integers.

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9 **Remark 1**

10 *It is obvious that if $\Omega' \geq \Omega$ and Theorem 1 is true for Ω , then it is also true for*
11 *Ω' . It follows that if x can be reconstructed by (7), then it can be reconstructed*
12 *by the similar expression with $h' \leq h$ and $t_k = k \cdot h'$.*

13
14 **Remark 2**

15 *In the literature on signal processing, a different notation in terms of hertz*
16 *(number of cycles per second) is often used. To explain the difference, we*
17 *remark that the angular frequency ω used by us corresponds to the frequency*
18 *in hertz f in such a way that $\omega = 2\pi \cdot f$. If we set up the boundary frequency*
19 *as $W = \frac{\Omega}{2\pi}$, we obtain that $h = \frac{\pi}{\Omega} = \frac{1}{2W}$. With this notation we reproduce the*
20 *Shannon's version of sampling theorem ([17]):*

21
22 If a function x contains no frequencies higher than W hertz, it is
23 completely determined by giving its ordinates at a series of points
24 spaced $\frac{1}{2W}$ seconds apart.

25
26 *The reconstruction formula (7) takes the form:*

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$$x(t) = \sum_{k=-\infty}^{\infty} x\left(\frac{k}{2W}\right) \cdot \text{sinc}(2Wt - k). \quad (9)$$

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31 *The value $2W$, for which W is the lowest bound of frequencies fulfilling the*
32 *sampling theorem, is called the Nyquist rate (see, e.g., [2]).*

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34 **Important Comment:** Due to the infinite number of samples and usage of
35 their exact values, reconstruction formula (7) is a theoretical tool only. In
36 practice, it is used for a finite number of measured values of a signal x . If a
37 measurement error has known statistical estimates (mean and deviation), then
38 the problem is to find a filter that can reduce an influence of such kind of noise.
39 Below, we will show that the technique of F-transform can be used as an efficient
40 filter.
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43 **3. The F-Transform: Short Reminder**

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45 The greatest drawback of the classical Fourier transform is a rather narrow
46 class of functions (originals) for which it can be effectively computed. Namely,
47 it is necessary that these functions decrease sufficiently rapidly to zero (in the
48 neighborhood of infinity) in order to insure the existence of the Fourier integral.
49 For example, the Fourier transform of such simple functions as polynomials does
50 not exist in the classical sense.
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52 *What is the F-transform.* The F-transform (originally, fuzzy transform) is an-
53 other type of integral transforms that is determined by a *fuzzy partition* of a
54 universe of discourse, say \mathbb{R} , and applied to functions from $L_2(\mathbb{R})$. We remind
55 that the name F-transform (short of *fuzzy transform*) was motivated by the
56 ideas and techniques of fuzzy logic (see, e.g., [5, 7, 20]). The notion of fuzzy
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partition has a certain evolution in fuzzy literature and especially in connection with the F-transform (see [4, 15, 19]). Below, we use a particular case of a generalized fuzzy partition [15], which is connected with (produced by) a generating function.

A non-negative continuous even and bell-shaped function $a : \mathbb{R} \rightarrow [0, 1]$ that vanishes outside $[-1, 1]$ (has a *compact support*) and fulfills $\int_{-1}^1 a(t) dt = 1$ is called a *generating function*. An example of generating function is the *raised cosine*:

$$a(t) = \begin{cases} \frac{1}{2}(1 + \cos(\pi t)), & -1 \leq t \leq 1, \\ 0, & \text{otherwise.} \end{cases} \quad (10)$$

A generating function a produces infinitely many rescaled functions $a_H : \mathbb{R} \rightarrow [0, 1]$ such that

$$a_H(t) \stackrel{\text{def}}{=} a\left(\frac{t}{H}\right),$$

where H is a positive number called a *scale factor*. It is easy to see that (substituting $s = \frac{t}{H}$)

$$\int_{-\infty}^{\infty} a_H(t) dt = \int_{-H}^H a_H(t) dt = \int_{-H}^H a\left(\frac{t}{H}\right) dt = H \cdot \int_{-1}^1 a(s) ds = H.$$

Let a be a generating function and $H > 0$ a scale factor. Let $t_k = t_0 + k \cdot h$, $k \in \mathbb{Z}$, be uniformly distributed nodes in \mathbb{R} , $t_0 \in \mathbb{R}$ and $h > 0$ ².

With each node t_k we correspond the translation $a_{H,k}(t) = a_H(t_k - t)$ of a_H , so that the set $\{a_{H,k}, k \in \mathbb{Z}\}$ establishes a (h, H) -uniform fuzzy partition of \mathbb{R} . We remind that a (h, H) -uniform fuzzy partition³ of \mathbb{R} with nodes $t_k = k \cdot h$, $k \in \mathbb{Z}$, is constituted by fuzzy sets $a_{H,k}$, $k \in \mathbb{Z}$, if $H > h/2$ and the following conditions are fulfilled:

1. (locality) – $a_{H,k}(t) > 0$ if $t \in (t_k - H, t_k + H)$ and $a_{H,k}(t) = 0$ if $t \in \mathbb{R} \setminus [t_k - h, t_k + H]$;
2. (continuity) – $a_{H,k}$ is continuous on $[t_k - H, t_k + H]$;
3. (covering) – for $t \in \mathbb{R}$, $\sum_{k=1}^n a_{H,k}(t) > 0$;
4. (uniformity) – fuzzy sets $a_{H,k}$, $k \in \mathbb{Z}$, are translations of a rescaled generating function $a : [-1, 1] \rightarrow [0, 1]$, i.e., $a_{H,k}(t) = a_H(t_k - t) = a\left(\frac{t_k - t}{H}\right)$.

If $h = H$, then a (h, H) -uniform fuzzy partition is called a h -uniform fuzzy partition.

²For simplicity of representation, we assume that $t_0 = 0$.

³The name “fuzzy partition” is used because of two reasons: at first, functions $a_{H,k}$ represent fuzzy sets and at second, after normalization these fuzzy sets can be considered as similarity classes of corresponding core points.

Assume that $x \in L_2(\mathbb{R})$ is a function. The sequence $(X_k), k \in Z$, where

$$X_k = \frac{\int_{-\infty}^{\infty} a_H(t_k - s) \cdot x(s) ds}{\int_{-\infty}^{\infty} a_H(s) ds} = \frac{1}{H} \int_{t_k-H}^{t_k+H} a_H(t_k - s) \cdot x(s) ds, \quad (11)$$

is the *F-transform* of x with respect to $\{a_{H,k}, k \in \mathbb{Z}\}$. $X_k, k \in Z$, are the *F-transform components* of x .

The basic idea of the F-transform is to capture a “local approach to data”⁴. It follows from (11) that different to the Fourier case, the F-transform can be effectively computed for a rather wide class of functions. Indeed, the latter are restricted by existence of integrals with finite ranges. In particular, polynomials can be originals of the F-transform.

4. Reconstruction from the F-transform Components

Usually, we lose information when we replace an original function (signal) by its F-transform. However, similarly to the case of the Nyquist-Shannon-Kotel’nikov reconstruction, additional properties of a signal (e.g., smoothness) allows its reconstruction even from the F-transform components.

Our main result: exact formulation. Similar to the original Nyquist-Shannon-Kotel’nikov theorem, we consider

- (a1) a signal x , for which all components $\widehat{x}(\omega)$ of frequency $|\omega| \geq \Omega = \frac{\pi}{h}$ are zeros,
- (a2) a generating function a such that

$$\widehat{a}(\omega) \neq 0 \text{ for all } \omega \in [-\Omega, \Omega].$$

Below, we will show that a continuous and band-limited signal x can be uniquely reconstructed from the set of its F-transform components and moreover, the sampling theorem can be obtained as a particular case.

Let us recall the “rectangular pulse” function $\mathbf{1}_{[-a,a]}$ (the characteristic function of $[-a, a]$)

$$\mathbf{1}_{[-a,a]}(t) = \begin{cases} 1, & \text{if } |t| \leq a; \\ 0, & \text{if } |t| > a, \end{cases}$$

and its Fourier transform

$$\widehat{\mathbf{1}_{[-a,a]}} = 2 \frac{\sin(a\omega)}{\omega}. \quad (12)$$

⁴Component X_k of the F-transform represents local data around the node t_k , where the latter is specified by corresponding generating function.

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9 **Theorem 2 (Reconstruction from the F-transform)**

10 Let signal $x \in L_2(\mathbb{R})$ be continuous and band-limited, i.e. $\widehat{x}(\omega) = 0$ for $|\omega| > \Omega$
 11 where Ω is some constant. Let $h = \frac{\pi}{\Omega}$, $a : \mathbb{R} \rightarrow [0, 1]$ be a generating function,
 12 and $H > h/2$ a scale factor. Let the set of translations $\{a_{H,k}, k \in \mathbb{Z}\}$, where
 13 $a_{H,k}(s) = a_H(t_k - s)$, establish a (h, H) -uniform fuzzy partition of \mathbb{R} with nodes
 14 $t_k = k \cdot h$, $k \in \mathbb{Z}$, so that the sequence $\{X_k, k \in \mathbb{Z}\}$ consists of the corresponding
 15 F-transform components of x . Let finally $\widehat{a_H}(\omega) \neq 0$, for all $\omega \in [-\Omega, \Omega]$.

16 Then x can be determined by its F-transform components so that

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$$x(t) = \frac{H\pi}{\Omega} \sum_{k=-\infty}^{\infty} X_k \cdot b_H(t - t_k), \quad (13)$$

20 where according to (11),

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$$X_k = X(t_k) = \frac{1}{H} \int_{-\infty}^{\infty} a_{H,k}(s) \cdot x(s) ds,$$

24 and $b_H \in L_2(\mathbb{R})$ is the function whose Fourier transform is equal to

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$$\widehat{b_H}(\omega) = \frac{\mathbf{1}_{[-\Omega, \Omega]}}{\widehat{a_H}(\omega)}. \quad (14)$$

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32 PROOF: Let us extend expression (11) to

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$$X(t) = \frac{\int_{-\infty}^{\infty} a_H(t - s) \cdot x(s) ds}{H}, \quad t \in \mathbb{R},$$

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36 so that $X : \mathbb{R} \rightarrow \mathbb{R}$ is a function such that $X(t_k) = X_k$, $k \in \mathbb{Z}$, where X_k is
 37 the k -th F-transform component of x with respect to the fuzzy partition given
 38 by $\{a_{H,k}, k \in \mathbb{Z}\}$. Notice that

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$$X = \frac{1}{H}(a_H * x),$$

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43 where $a_H * x$ is a convolution of a_H and x . By the assumptions, $a_H \in L_1(\mathbb{R})$
 44 and $x \in L_2(\mathbb{R})$, so that $(a_H * x) \in L_2(\mathbb{R})$ and thus, $X \in L_2(\mathbb{R})$. Moreover, X
 45 is continuous, because it is a convolution of a_H that has a compact support and
 46 x that is locally integrable on \mathbb{R} (follows from continuity of x on \mathbb{R}).
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48 Therefore, by the properties of the Fourier transform, X can be represented
 49 by the inversion formula

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51
$$X(t) = \text{l.i.m.}_{n \rightarrow \infty} \frac{1}{2\pi} \int_{-n}^n \widehat{X}(\omega) e^{i\omega t} d\omega, \quad (15)$$

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53 where (by convolution-to-product theorems (see e.g., [1]))

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$$\widehat{X}(\omega) = \frac{\widehat{x}(\omega) \cdot \widehat{a_H}(\omega)}{H}. \quad (16)$$

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9 It follows that \widehat{X} is band-limited and $\widehat{X}(\omega) = 0$ for $|\omega| > \Omega$. Therefore, by (15)
10 and continuity of X , we have the exact representation

$$11 \quad X(t) = \frac{1}{2\pi} \int_{-\Omega}^{\Omega} \widehat{X}(\omega) e^{i\omega t} d\omega. \quad (17)$$

12
13
14 Because $\widehat{X} \in L_2[-\Omega, \Omega]$, it can be expanded in a Fourier series

$$15 \quad \widehat{X}(\omega) = \sum_{k=-\infty}^{\infty} g_k e^{-ik\pi\omega/\Omega} \quad (18)$$

16
17 where

$$18 \quad g_k = \frac{1}{2\Omega} \int_{-\Omega}^{\Omega} \widehat{X}(\omega) e^{ik\pi\omega/\Omega} d\omega.$$

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20 By (17),

$$21 \quad g_k = \frac{\pi}{\Omega} X\left(\frac{k\pi}{\Omega}\right) = \frac{\pi}{\Omega} X(t_k) = \frac{\pi}{\Omega} X_k,$$

22
23 where X_k is the F-transform component of X with respect to $\{a_{H,k}, k \in \mathbb{Z}\}$.
24 Substituting g_k into (18), we get

$$25 \quad \widehat{X}(\omega) = \frac{\pi}{\Omega} \sum_{k=-\infty}^{\infty} X_k e^{-it_k\omega}. \quad (19)$$

26
27 Because signal x fulfills the assumptions of Theorem 1, we can express it
28 with the help of the inversion formula

$$29 \quad x(t) = \frac{1}{2\pi} \int_{-\Omega}^{\Omega} \widehat{x}(\omega) e^{i\omega t} d\omega,$$

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31 where by (16),

$$32 \quad \widehat{x}(\omega) = \frac{H \cdot \widehat{X}(\omega)}{\widehat{a}_H(\omega)}.$$

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34 Hence, we have

$$35 \quad x(t) = \frac{H}{2\pi} \int_{-\Omega}^{\Omega} \frac{\widehat{X}(\omega)}{\widehat{a}_H(\omega)} e^{i\omega t} d\omega,$$

36
37 and after substituting $\widehat{X}(\omega)$ from (19)

$$38 \quad x(t) = \frac{H}{2\Omega} \sum_{k=-\infty}^{\infty} X_k \cdot \int_{-\Omega}^{\Omega} \frac{1}{\widehat{a}_H(\omega)} e^{i\omega(t-t_k)} d\omega. \quad (20)$$

39
40 Because \widehat{a}_H is continuous (as the inverse Fourier transform of a function from
41 $L_1(\mathbb{R})$) and $\widehat{a}_H(\omega) \neq 0$ in $[-\Omega, \Omega]$, the integral in the right-hand side of (20)
42 exists for all $k \in \mathbb{Z}$. Therefore, equality (20) proves the first claim of the theorem,
43 i.e. that x can be determined by the set of F-transform components.
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In order to prove the second claim, let us consider the integral in the right-hand side of (20)

$$\int_{-\Omega}^{\Omega} \frac{1}{\widehat{a}_H(\omega)} e^{i\omega(t-t_k)} d\omega = \int_{-\infty}^{\infty} \frac{\mathbf{1}_{[-\Omega, \Omega]}}{\widehat{a}_H(\omega)} e^{i\omega(t-t_k)} d\omega.$$

The function $\frac{\mathbf{1}_{[-\Omega, \Omega]}}{\widehat{a}_H(\omega)}$ has a compact support and thence belongs to $L_1(\mathbb{R}) \cap L_2(\mathbb{R})$. Therefore by the inversion theorem for L_2 functions, there is a continuous function, say $b_H \in L_2(\mathbb{R})$, such that $\widehat{b}_H(\omega) = \frac{\mathbf{1}_{[-\Omega, \Omega]}}{\widehat{a}_H(\omega)}$ and

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\mathbf{1}_{[-\Omega, \Omega]}}{\widehat{a}_H(\omega)} e^{i\omega(t-t_k)} d\omega = b_H(t-t_k).$$

As a last step we obtain (13) from (20), i.e.

$$x(t) = \frac{H\pi}{\Omega} \sum_{k=-\infty}^{\infty} X_k \cdot b_H(t-t_k).$$

□

In the below given Corollary 2, we give another expression for reconstruction formula (13), which includes the function whose Fourier transform is equal to the reversed generating function a .

Corollary 2

Let signal x fulfil the assumptions of Theorem 2. Then x can be reconstructed from its F -transform components so that

$$x(t) = \frac{h}{H} \sum_{k=-\infty}^{\infty} X_k \cdot b\left(\frac{t-t_k}{H}\right), \quad (21)$$

where $b \in L_2(\mathbb{R})$ is the function whose Fourier transform is equal to

$$\widehat{b}(\omega) = \frac{\mathbf{1}_{[-H\Omega, H\Omega]}}{\widehat{a}(\omega)}. \quad (22)$$

Moreover,

$$b_H(t) = \frac{1}{H^2} \cdot b\left(\frac{t}{H}\right),$$

where b_H is used in reconstruction formula (13).

PROOF: Because $a_H(t) = a\left(\frac{t}{H}\right)$, we have that $\widehat{a}_h(\omega) = H\widehat{a}(H\omega)$. Therefore,

$$\widehat{b}_H(\omega) = \frac{\mathbf{1}_{[-\Omega, \Omega]}}{\widehat{a}_H(\omega)} = \frac{\mathbf{1}_{[-\Omega, \Omega]}}{H\widehat{a}(H\omega)}.$$

After substitution $H\omega = \omega'$ we obtain

$$\widehat{b}_H\left(\frac{\omega'}{H}\right) = \frac{\mathbf{1}_{[-\Omega H, \Omega H]}}{H\widehat{a}(\omega')} = \frac{1}{H}\widehat{b}(\omega'),$$

or

$$\widehat{b}_H(\omega) = \frac{1}{H^2} \cdot H\widehat{b}(H\omega).$$

It easily follows that

$$b_H(t) = \frac{1}{H^2} \cdot b\left(\frac{t}{H}\right).$$

To prove (21) it is sufficient to substitute the expression above into (13) and use that $h = \frac{\pi}{\Omega}$.

$$\begin{aligned} x(t) &= \frac{H\pi}{\Omega} \sum_{k=-\infty}^{\infty} X_k \cdot b_H(t - t_k) = \frac{H\pi}{\Omega} \cdot \frac{1}{H^2} \sum_{k=-\infty}^{\infty} X_k \cdot b\left(\frac{t - t_k}{H}\right) = \\ &= \frac{h}{H} \sum_{k=-\infty}^{\infty} X_k \cdot b\left(\frac{t - t_k}{H}\right). \end{aligned}$$

□

Remark 3

If in (21) we assume that $H = h$, then the reconstruction from the F-transform components takes the form

$$x(t) = \sum_{k=-\infty}^{\infty} X_k \cdot b\left(\frac{t - t_k}{h}\right) = \sum_{k=-\infty}^{\infty} X_k \cdot b\left(\frac{t}{h} - k\right), \quad (23)$$

where $b \in L_2(\mathbb{R})$ is the function whose Fourier transform is equal to

$$\widehat{b}(\omega) = \frac{\mathbf{1}_{[-\pi, \pi]}}{\widehat{a}(\omega)}. \quad (24)$$

Reconstruction (23) is similar to the Nyquist-Shannon-Kotel'nikov formula given by (7). To illustrate this similarity and on the other side, show the difference between two kernel functions *sinc* and *b*, determined by (8) and (24), respectively, we put in Fig. 1 plots of both of them.

Corollary 3

Let the assumptions of Theorem 2 be fulfilled for two signals x and y . Let moreover, for all $k \in \mathbb{Z}$, x and y have equal F-transform components computed for a (h, H) -uniform fuzzy partition of \mathbb{R} with nodes $t_k = k \cdot h$, $k \in \mathbb{Z}$. Then the signals coincide on the whole domain, i.e.

$$x(t) = y(t), \quad t \in \mathbb{R}.$$

Corollary 4

Let signal x fulfil the assumptions of Theorem 2, and the (Dirac) delta function δ be chosen instead of generating function a . Let moreover, $H = h$ and a generalized h -uniform fuzzy partition of \mathbb{R} be given by the set of translations

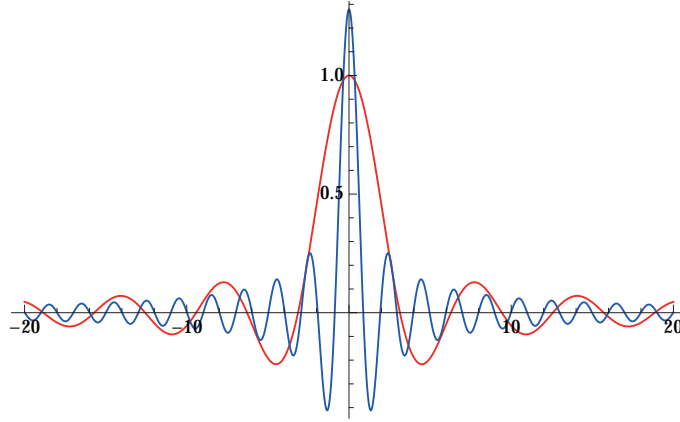


Figure 1: Plots of kernel functions sinc (in red) and b (in blue), that are used in the Nyquist-Shannon-Kotel'nikov and the F-transform based reconstructions, respectively.

$\{\delta_k, k \in \mathbb{Z}\}$ where $\delta_k(t) = \delta(t_k - t)$. Then reconstruction formula (23) where $b \in L_2(\mathbb{R})$ is the function whose Fourier transform is equal to

$$\widehat{b}(\omega) = \frac{\mathbf{1}_{[-\pi, \pi]}}{\widehat{\delta}(\omega)}, \quad (25)$$

reduces to the original Nyquist-Shannon-Kotel'nikov reconstruction in the form of (7).

PROOF: To prove the claim, it is sufficient to find a representation of function b such that its Fourier transform is equal to (25). It is known that $\widehat{\delta}(\omega) = 1$, and therefore,

$$\widehat{b}(\omega) = \mathbf{1}_{[-\pi, \pi]}.$$

Thus,

$$b(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\omega t} d\omega = \frac{1}{\pi t} \sin(\pi t) = \text{sinc}(t).$$

Let us substitute the last expression for b into (23) and obtain that

$$x(t) = \sum_{k=-\infty}^{\infty} X_k \cdot \text{sinc}\left(\frac{t - t_k}{h}\right).$$

Finally by (11),

$$X_k = \frac{\int_{-\infty}^{\infty} \delta(t_k - s) \cdot x(s) ds}{\int_{-\infty}^{\infty} \delta(s) ds} = x(t_k), \quad k \in \mathbb{Z},$$

so that reconstruction (23) coincides with (7). \square

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5. Effect of Measurement Errors on Signal Reconstruction

In this section, we will pay attention to the case where sample values of a signal come with noise. We will show that in the presence of noise, a more accurate reconstruction than that based on the sampling theorem can be obtained, if instead of noised sample values the F-transform components of the signal with respect to a generalized fuzzy partition is used.

We assume that continuous and band-limited signal $x \in L_2(\mathbb{R})$ is given by its approximately measured sample values $\tilde{x}(t_k)$ at points $t_k = k \cdot h$, $k \in \mathbb{Z}$, where $h = \frac{\pi}{\Omega}$ and Ω is the upper band limit. In this case, “reconstruction” by (7), where $\tilde{x}(t_k)$ are substituted for $x(t_k)$, $k \in \mathbb{Z}$, gives a new signal $\tilde{x}(t)$

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} \tilde{x}(t_k) \cdot \text{sinc} \left(\frac{t}{h} - k \right), \quad (26)$$

which differs from x .

In this section, we will estimate statistical characteristics of reconstruction (26) and discuss the problem whether it is possible to reduce the influence of measurement errors at points $t \in \mathbb{R}$ different from t_k , $k \in \mathbb{Z}$.

We will be focused on the following estimation of the measurement error function

$$\Delta x(t) \stackrel{\text{def}}{=} \tilde{x}(t) - x(t),$$

which can be formally expressed from the above as

$$\Delta x(t) = \sum_{k=-\infty}^{\infty} \Delta x(t_k) \cdot \text{sinc} \left(\frac{t}{h} - k \right). \quad (27)$$

In what follows, we assume that

(b1) for any k , the measurement errors $\Delta x(t_k)$ are normally distributed random variables with zero mean values and finite variances ε^2 , i.e., for all $k \in \mathbb{Z}$, $\Delta x(t_k) \sim N(0, \sigma^2)$;

(b2) random variables $\Delta x(t_k)$ and $\Delta x(t_\ell)$ are independent, if $k \neq \ell$.

Remark 4

Let us remark that “almost all” series in the right-hand side of (26) (similarly, (27)) are convergent, i.e. the probability that the series in the right-hand side of (26) does not converge is equal to zero. This fact is a simple consequence of the assumption about finite variance of normally distributed random variables $\Delta x(t_k)$, $k \in \mathbb{Z}$.

5.1. Statistical characteristics of measurement errors by the Nyquist-Shannon-Kotel’nikov reconstruction

In this section, we will show that if signal \tilde{x} is reconstructed by (26), where the approximately measured values $\tilde{x}(t_k)$ are used instead of actual ones, then

the main statistical characteristics (mean and variance) of the measurement error function (27) at any point $t \in \mathbb{R}$ remain unchanged. In other words, the Nyquist-Shannon-Kotel'nikov reconstruction (27) reproduces statistical characteristics of random variables $\Delta x(t_k)$, $k \in \mathbb{Z}$.

Theorem 3

Let a signal $x \in L_2(\mathbb{R})$ be continuous and band-limited, i.e. $\widehat{x}(\omega) = 0$ for $|\omega| > \Omega$. Denote $h = \frac{\pi}{\Omega}$ and $t_k = k \cdot h$, $k \in \mathbb{Z}$. Assume that for any $k \in \mathbb{Z}$, the error $\Delta x(t_k)$ of every measured value $\tilde{x}(t_k)$ of the sample $x(t_k)$ is a random variable such that (b1) and (b2) are fulfilled. Then, for each $t \in \mathbb{R}$, the random variable $\Delta x(t)$, defined by (27), is normally distributed with the zero mean and variance $\sigma^2(t)$ for which

$$\sigma^2(t) = \varepsilon^2. \tag{28}$$

PROOF: Let $\Delta x(t_k) \sim N(0, \varepsilon^2)$ where $t_k = k \cdot h$, $k \in \mathbb{Z}$, i.e., $\Delta x(t_k)$ is normally distributed with the mean 0 and variance ε^2 . Obviously, for arbitrary $N > 0$, we have

$$\Delta x_N(t) = \sum_{k=-N}^N \Delta x(t_k) \cdot \text{sinc}\left(\frac{t}{h} - k\right) \sim N\left(0, \sum_{k=-N}^N \varepsilon^2 \cdot \text{sinc}^2\left(\frac{t}{h} - k\right)\right).$$

Since $\Delta x(t)$ is the limit case of $\Delta x_N(t)$, where $N \rightarrow \infty$, we obtain

$$\Delta x(t) = \sum_{k=-\infty}^{\infty} \Delta x(t_k) \cdot \text{sinc}\left(\frac{t}{h} - k\right) \sim N\left(0, \sum_{k=-\infty}^{\infty} \varepsilon^2 \cdot \text{sinc}^2\left(\frac{t}{h} - k\right)\right).$$

Therefore, the measurement error $\Delta x(t)$ is normally distributed with the zero mean and the variance expressed in the form

$$\sigma^2(t) = \varepsilon^2 \cdot \sum_{k=-\infty}^{\infty} \text{sinc}^2\left(\frac{t}{h} - k\right) = \varepsilon^2 \cdot \sum_{k=-\infty}^{\infty} \text{sinc}^2\left(\frac{t}{h} - k\right). \tag{29}$$

Let us verify (28), i.e. for every t , $\sigma^2(t) = \varepsilon^2$.

To do this, we show that the sum in the right-hand side of (29) is the sum of Fourier coefficients for $f(t) = e^{iat/h} \mathbf{1}_{[-\pi, \pi]}$ and then apply the Parseval's identity. Indeed, the k -th Fourier coefficient for f is equal to

$$\widehat{f}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt} dt = \frac{1}{\pi} \cdot \frac{\sin(\pi(k - a/h))}{k - a/h} = \text{sinc}\left(k - \frac{a}{h}\right),$$

where we made use of (12). Recall that the Parseval's identity for $f \in L_2[-\pi, \pi]$ is as follows:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t)|^2 dt = \sum_{k=-\infty}^{\infty} |\widehat{f}(k)|^2,$$

where $|\cdot|$ is an absolute value of a complex number. Because

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t)|^2 dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{iat/h} e^{-iat/h} dt = 1,$$

we come to the Parseval's identity in the form of the equality

$$1 = \sum_{k=-\infty}^{\infty} \operatorname{sinc}^2\left(k - \frac{a}{h}\right).$$

The latter confirms (28). \square

5.2. Reduced influence of measurement errors by the reconstruction from the F-transform components

In this section, we again assume that samples $x(t_k)$, $k \in \mathbb{Z}$, are obtained with measurement errors, i.e. we get approximately measured values $\tilde{x}(t_k)$ instead of actual ones. We will see that opposite to the direct reconstruction (26) by the Nyquist-Shannon-Kotel'nikov formula, the reconstruction from the F-transform components with the specially adjusted parameters h and H reduces the influence of measurement errors. In order to achieve this goal we propose to choose the scale factor H with the borderline value $H = \frac{\pi}{\Omega}$ and the sample step $h \ll H$. Our detailed assumptions are as follows:

- signal $x \in L_2(\mathbb{R})$ is continuous and band-limited, i.e. $\hat{x}(\omega) = 0$ for $|\omega| > \Omega$, where Ω is some constant;
- $H = \frac{\pi}{\Omega}$, $a : \mathbb{R} \rightarrow [0, 1]$ is a generating function, and the set of translations $\{a_{H,\ell}, \ell \in \mathbb{Z}\}$, where $a_{H,\ell}(s) = a_H(T_\ell - s)$, establish a *H-uniform fuzzy partition* of \mathbb{R} with nodes $T_\ell = H \cdot \ell$, $\ell \in \mathbb{Z}$;
- the sequence $\{X_\ell, \ell \in \mathbb{Z}\}$ consists of the corresponding F-transform components of x , where

$$X_\ell = \frac{1}{H} \int_{H_\ell - H}^{T_\ell + H} a_H(H_\ell - s) \cdot x(s) ds.$$

- at each node T_ℓ , X_ℓ is approximated by the rectangular rule

$$X_\ell \approx \frac{h}{H} \sum_{k=-\infty}^{\infty} a_H(T_\ell - t_k) \cdot x(t_k), \quad (30)$$

where $t_k = k \cdot h$, $k \in \mathbb{Z}$, are the sample points, at which signal x is known,

- sample step h divides the scale factor H .

We will proceed as follows: at first, we estimate the influence of measurement errors in the computation of F-transform components (Theorem 4) and then we apply Theorem 2 to show how the influence of measurement errors can be reduced in the computation (reconstruction) of signal values (Theorem 5). We will make some preliminary estimates.

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9 **Lemma 1**

10 Let $a : \mathbb{R} \rightarrow [0, 1]$ be a generating function so that it is continuous, even, bell-
11 shaped, vanishes outside $[-1, 1]$ and fulfills $\int_{-1}^1 a(t) dt = 1$. Then the following
12 estimate holds:

$$13 \quad \frac{1}{2} \leq \|a\|^2 \leq 1,$$

14 where the norm $\|\cdot\|$ is considered in $L_2([-1, 1])$.

15 PROOF: Obviously, $a \in L_2([-1, 1])$. Since for any $t \in [-1, 1]$, $a^2(t) \leq a(t)$, we
16 easily have the right-hand inequality $R(a) \leq 1$. The left-hand one follows from
17 the Cauchy - Schwarz inequality, i.e. from

$$18 \quad \left(\int_{-1}^1 a(t) dt \right)^2 \leq 2 \int_{-1}^1 a^2(t) dt.$$

19 □

20 For example, if a generating function is equal to the “raised cosine” (10), then
21 $R(a) = \frac{3}{4}$.

22 The below given theorem estimates the influence of measurement errors in
23 the computation of F-transform components X_ℓ , $\ell \in \mathbb{Z}$, given by (30). It shows
24 that the variance of the error distribution of ΔX_ℓ is smaller than that of $\Delta x(t_k)$.

25 **Theorem 4**

26 Let the assumptions above be fulfilled and moreover, signal x be sampled by
27 $x(t_k)$ at $t_k = k \cdot h$, where $h > 0$. We assume that each measured value $\tilde{x}(t_k)$ of
28 the sample $x(t_k)$ has error $\Delta x(t_k)$ that is assumed to be a random variable such
29 that (b1) and (b2) are satisfied. Let moreover, the F-transform components
30 X_ℓ , $\ell \in \mathbb{Z}$, be computed on the assumption (30) where the samples $x(t_k)$ are
31 replaced by values $\tilde{x}(t_k)$, $k \in \mathbb{Z}$, i.e. using the expression

$$32 \quad \tilde{X}_\ell = \frac{h}{H} \sum_{k=-\infty}^{\infty} a_H(T_\ell - t_k) \cdot \tilde{x}(t_k). \quad (31)$$

33 Then for every $\ell \in \mathbb{Z}$, the difference ΔX_ℓ

$$34 \quad \Delta X_\ell = X_\ell - \tilde{X}_\ell = \frac{h}{H} \sum_{k=-\infty}^{\infty} a_H(T_\ell - t_k) \cdot \Delta x(t_k), \quad (32)$$

35 is a normally distributed random variable with the zero mean and variance σ_ℓ^2
36 such that

$$37 \quad \sigma_\ell^2 \leq \frac{3h}{2H} \varepsilon^2, \quad (33)$$

38 where ε^2 is the variance of $\Delta x(t_k)$.

39 PROOF: The difference ΔX_ℓ , $\ell \in \mathbb{Z}$, is expressed by a linear combination of
40 independent normally distributed random variables $\Delta x(t_k)$. By the assumption

(b1), $\Delta x(t_k)$ is normally distributed, with mean 0 and variance ε^2 . Therefore, the variance σ_ℓ^2 of ΔX_ℓ is equal to

$$\sigma_\ell^2 = \frac{\varepsilon^2 \cdot h^2}{H^2} \sum_{k=-\infty}^{\infty} a_H^2(T_\ell - t_k).$$

Let us consider $h \sum_{k=-\infty}^{\infty} a_H^2(T_\ell - t_k)$ as a Riemann sum of the integral $\int_{T_\ell-H}^{T_\ell+H} a_H^2(T_\ell - s) ds$. Because h divides H and a has a bell shape, we have the estimate

$$h \sum_k a_H^2(T_\ell - t_k) - h \leq \int_{T_\ell-H}^{T_\ell+H} a_H^2(T_\ell - s) ds = H \int_{-1}^1 a^2(s) ds = H \|a\|^2.$$

It follows that

$$\sigma_\ell^2 \leq \frac{\varepsilon^2 \cdot h}{H} \left(\frac{h}{H} + \|a\|^2 \right)$$

Because $h \ll H$ and h divides H , we have that $\frac{h}{H} \leq \frac{1}{2}$. By this fact and Lemma 1, we finally obtain that

$$\sigma_\ell^2 \leq \frac{3h}{2H} \varepsilon^2.$$

□

In the below given theorem, we show that the variance $\sigma^2(t)$ of error distribution of $\Delta x(t)$ of signal x is smaller than that of $\Delta x(t_k)$, if the signal is reconstructed from its F-transform components. Moreover, we give an upper estimation of $\sigma^2(t)$, which includes the ratio $\frac{h}{H}$ between the borderline value $H = \frac{\pi}{\Omega}$ and the sample step h . This upper estimation shows how to make $\sigma^2(t)$ smaller.

Theorem 5

Let the assumptions of Theorem 4 be fulfilled, and signal \tilde{x} be reconstructed by (23) from the F-transform components \tilde{X}_ℓ , $\ell \in \mathbb{Z}$, i.e.

$$\tilde{x}(t) = \sum_{\ell=-\infty}^{\infty} \tilde{X}_\ell \cdot b\left(\frac{t - T_\ell}{H}\right), \quad (34)$$

where \tilde{X}_ℓ , $\ell \in \mathbb{Z}$, are computed by (31) using the measured values $\tilde{x}(t_k)$ instead of samples $x(t_k)$, $k \in \mathbb{Z}$, and $b \in L_2(\mathbb{R})$ is the function whose Fourier transform is equal to $\hat{b}(\omega) = \frac{1_{[-\pi, \pi]}}{\hat{a}(\omega)}$. Let similarly, original signal $x(t)$ be reconstructed by (23) from the F-transform components X_ℓ , $\ell \in \mathbb{Z}$, that are given by (30).

Then the difference $\Delta x(t) = x(t) - \tilde{x}(t)$, $t \in \mathbb{R}$, that is formally expressed by

$$\Delta x(t) = \sum_{\ell=-\infty}^{\infty} \Delta X_\ell \cdot b\left(\frac{t - T_\ell}{H}\right), \quad (35)$$

where ΔX_ℓ is given by (32), is a random variable with the zero mean and variance $\sigma^2(t)$. Moreover,

$$\sigma^2(t) \leq \frac{3h}{4H} \cdot \frac{\|1/\widehat{a}\|^2}{\pi} \varepsilon^2, \quad (36)$$

where the norm $\|\cdot\|$ is considered in $L_2([-\pi, \pi])$.

PROOF: For each $t \in \mathbb{R}$, the difference $\Delta x(t)$ in (35) is expressed by a linear combination of independent normally distributed random variables ΔX_ℓ . Therefore, $\Delta x(t)$ is a random variable too, so that

$$\Delta x(t) \sim N \left(0, \sum_{\ell=-\infty}^{\infty} \sigma_\ell^2 \cdot b^2 \left(\frac{t - T_\ell}{H} \right) \right). \quad (37)$$

By (37), $\Delta x(t)$, $t \in \mathbb{R}$, has the zero mean and variance $\sigma^2(t)$, such that

$$\sigma^2(t) = \sum_{\ell=-\infty}^{\infty} \sigma_\ell^2 \cdot b^2 \left(\frac{t - T_\ell}{H} \right).$$

Let us prove the estimate (36). By (33),

$$\sigma_\ell^2 \leq \frac{3h}{2H} \varepsilon^2,$$

and therefore,

$$\sigma^2(t) \leq \frac{3h}{2H} \varepsilon^2 \sum_{\ell=-\infty}^{\infty} b^2 \left(\frac{t - T_\ell}{H} \right). \quad (38)$$

Let us estimate the sum

$$\sum_{\ell=-\infty}^{\infty} b^2 \left(\frac{t - T_\ell}{H} \right),$$

where

$$b \left(\frac{t - T_\ell}{H} \right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{\widehat{a}(\omega)} e^{i\omega(t - T_\ell)/H} d\omega.$$

We use the fact that $T_\ell = H \cdot \ell$ (see the second item of detailed assumptions on page 15) and rewrite

$$b \left(\frac{t}{H} - \ell \right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\omega t/H}}{\widehat{a}(\omega)} e^{-i\omega \ell} d\omega.$$

If we fix the value of t in the above given equality, then its right-hand side can be considered as the finite Fourier transform

$$\widehat{f}(\ell) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{its/H}}{\widehat{a}(s)} e^{-i\ell s} ds, \ell \in \mathbb{Z},$$

of the function $f(s) = \frac{e^{its/H}}{\widehat{a}(s)} \in L_2[-\pi, \pi]$. By the Parseval's identity,

$$\sum_{\ell=-\infty}^{\infty} |\widehat{f}(\ell)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(s)|^2 ds.$$

It is easy to show that

$$\int_{-\pi}^{\pi} |f(s)|^2 ds = \int_{-\pi}^{\pi} \left| \frac{e^{its/H}}{\widehat{a}(s)} \right|^2 ds = \int_{-\pi}^{\pi} \left| \frac{1}{\widehat{a}(s)} \right|^2 ds = \|1/\widehat{a}\|^2.$$

Therefore,

$$\sum_{\ell=-\infty}^{\infty} b^2 \left(\frac{t - T_\ell}{H} \right) = \frac{1}{2\pi} \|1/\widehat{a}\|^2.$$

Finally, we substitute the obtained right-hand side into (38) and come to the desired estimate

$$\sigma^2(t) \leq \frac{3h}{2H} \varepsilon^2 \sum_{\ell=-\infty}^{\infty} b^2 \left(\frac{t - T_\ell}{H} \right) = \frac{3h}{4H} \cdot \frac{\|1/\widehat{a}\|^2}{\pi} \varepsilon^2.$$

□

5.3. Experiments and Illustrations

We chose $x(t) = \sin(t) + \sin(2t)$ as a continuous and band-limited signal and considered it on the interval $[-20, 20]$. The upper band limit $\Omega = 2.1$ was chosen to guarantee that $\widehat{x}(\omega) = 0$ for $|\omega| \geq \Omega$. In Fig. 2, the graph of $x(t)$ is given.

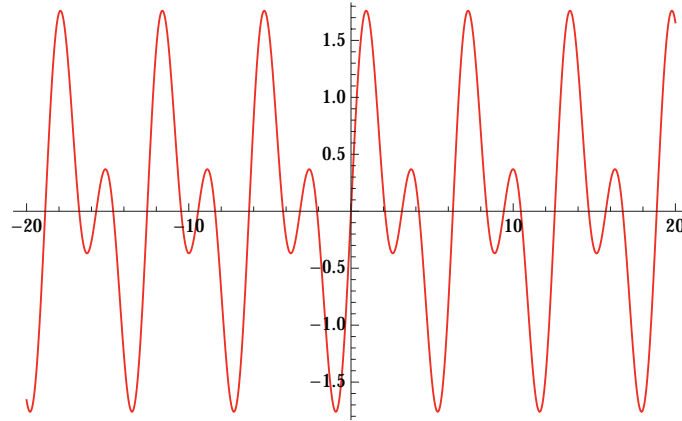


Figure 2: Continuous and band-limited signal $x(t) = \sin(t) + \sin(2t)$.

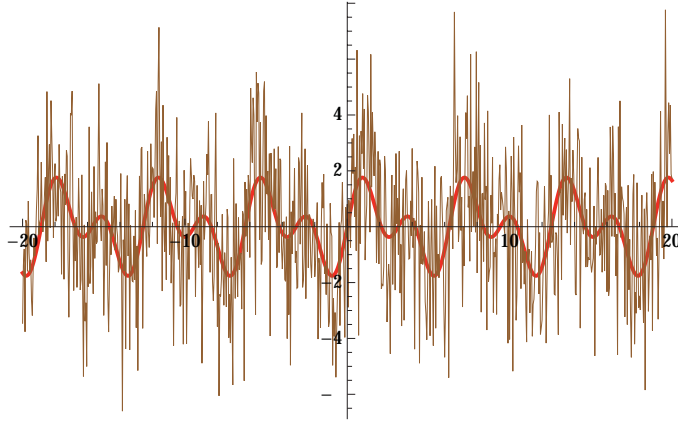


Figure 3: Original signal $x(t) = \sin(t) + \sin(2t)$ (in red) and noised signal $\tilde{x}(t) = \sin(t) + \sin(2t) + N(0, 4)$ (in brown), where $N(0, 4)$ is the Gaussian function with the mean 0 and deviation 2.

We contaminated the signal x by the Gaussian noise with deviation $\sigma = 2$. In Fig. 3, both graphs of the original signal $x(t)$ and the noised signal $\tilde{x}(t)$ are given.

We carried out two reconstructions from the noised sample values $\tilde{x}(t_k)$, $t_k = k \cdot h$, $k \in \mathbb{Z}$: the one was based on the Nyquist-Shannon-Kotel'nikov formula (7), and the other one on the F-transform components (34). The following parameters were chosen: $H = \frac{\pi}{\Omega}$ ($\Omega = 2.1$) as a scale factor, and $h = 0.05$ as a step value of sample points. These parameters were used in the Nyquist-Shannon-Kotel'nikov formula (7) and in the computation by (31) of the corresponding F-transform components of \tilde{x} . The F-transform was specified by the generating function (10) (raised cosine) and the corresponding H -uniform fuzzy partition of \mathbb{R} with nodes $T_\ell = H \cdot \ell$, $\ell \in \mathbb{Z}$. The result is illustrated in Fig. 4. The illustration shows that the reconstruction from F-transform components significantly diminishes the applied type of noise.

The last experiment relates to the same case of a signal and noise as above. The only difference was made in the choice of the step value h , which has been chosen smaller than above, i.e. $h = 0.001$. We expected that after the reconstruction from F-transform components the variance of an error distribution of the noised signal would be also smaller (in comparison with the preceding experiment). The result in Fig. 5 confirms this expectation. Moreover, it is obvious that the reconstruction from F-transform components successfully removes this type of noise.

6. Conclusion

The paper contributes to the theory of F-transforms as well as to the theory of signals. At first, the problem of reconstruction from a set of samples

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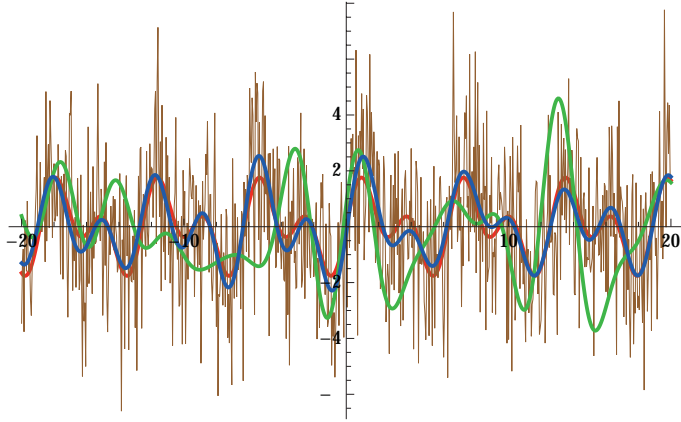


Figure 4: Original signal $x(t) = \sin(t) + \sin(2t)$ (in red), noisy signal $\tilde{x}(t) = \sin(t) + \sin(2t) + N(0, 4)$ (in brown), reconstruction by the Nyquist-Shannon-Kotel'nikov formula (in green), reconstruction from the F-transform components (in blue). The reconstructions are performed with parameters $H = \frac{\pi}{2.1}$ and $h = 0.05$.

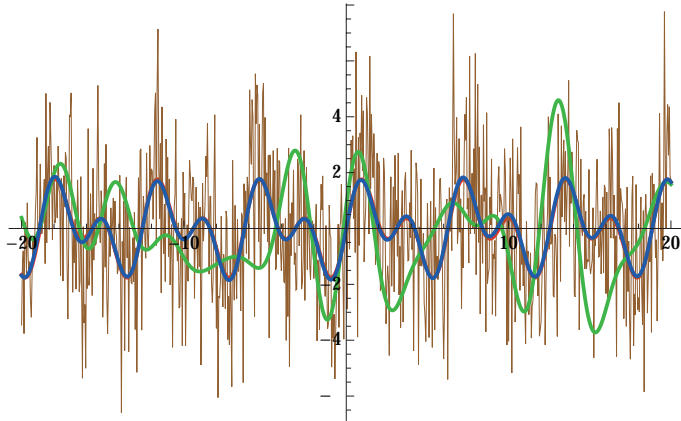


Figure 5: Original signal $x(t) = \sin(t) + \sin(2t)$ (in red), noisy signal $\tilde{x}(t) = \sin(t) + \sin(2t) + N(0, 4)$ (in brown), reconstruction by the Nyquist-Shannon-Kotel'nikov formula (in green), reconstruction from the F-transform components (in blue). The reconstructions are performed with parameters $H = \frac{\pi}{2.1}$ and $h = 0.001$. The red curve (original signal) almost coincides with the blue one (reconstruction from the F-transform components).

was considered under the conventional assumptions on a signal: continuity and band-limitation. We showed that any such signal can be reconstructed from a countable set of its F-transform components. We showed that the classical result that is known as the Nyquist-Shannon-Kotel'nikov reconstruction is a particular case of the reconstruction from the F-transform components. In the

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9 correspondence with the theory of F-transforms, this result shows that despite
10 of the fact that the inverse F-transform is lossy, an original function can be
11 recovered from a sequence of its F-transform components.

12 At second, we considered the case where sample values of a signal come with
13 noise. We show that in the presence of noise, a more accurate reconstruction
14 than that based on the sampling theorem can be obtained, if instead of noised
15 sample values the F-transform components of the signal are used. Moreover,
16 we showed that a proper choice of a fuzzy partition and a sample step leads to
17 a significant reduction of a noise. We obtained a theoretical upper estimation
18 of the variance $\sigma^2(t)$ of error distribution of $\Delta x(t)$ of signal x that includes
19 the ratio $\frac{h}{H}$ between the value of H , which characterizes a H -uniform fuzzy
20 partition, and the sample step h . This upper estimation shows how to make
21 $\sigma^2(t)$ smaller. We included experiments that illustrate the achieved results.
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