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Why Lattice-Valued Fuzzy Values? A Mathematical Justification

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Abstract. To take into account that expert's degrees of certainty are not always comparable, researchers have used partially ordered set of degrees instead of the more traditional linearly (totally) ordered interval $[0, 1]$. In most cases, it is assumed that this partially ordered set is a lattice, i.e., every two elements have the greatest lower bound and the least upper bound. In this paper, we prove a theorem explaining why it is reasonable to require that the set of degrees is a lattice.

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1. Formulation of the Problem

Traditional $[0, 1]$*-based fuzzy logic: a brief reminder.* To describe the expert's degree of certainty about different statements, Lofti A. Zadeh originally proposed to use numbers from the interval $[0, 1]$ [4], so that:

- the value 1 indicates full certainty in the statement,
- the value 0 indicates full certainty that the statement is false, and
- intermediate values describe intermediate degrees of certainty.

Need to go beyond the interval $[0, 1]$. Numbers from the interval $[0, 1]$ are totally (linearly) ordered: for every two numbers a and b, we either have $a \leq b$ or $b \leq a$. Thus, this representation implicitly assumes that we can always compare our degrees of confidence and decide which one corresponds to larger confidence.

In reality, we sometimes have incomparable degree of confidence, for which neither the first not the second one corresponds to higher confidence. To capture such situations, Zadeh proposed to use, as a set of possible degrees of confidence, a *partially ordered* set, in which there may exists elements a and b for which

$$a \not\leq b \text{ and } b \not\leq a.$$

Lattices are frequently used. Different partially ordered sets have been used to describe experts' degrees of confidence. Most frequently, *lattices* are used, i.e., partially ordered sets for which, for every two elements a and b, there exist two special elements:

- the smallest of all elements c which are larger than or equal to both a and b; this smallest element is known as the *least upper bound*, or *join* of a and b;
- the largest of all elements c which are smaller than or equal to both a and b; this largest element is known as the *greatest lower bound*, or *meet* of a and b.

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In precise terms, the join is an element j for which:

- first, $a \le j$ and $b \le j$, and
- second, for every element c for which $a \le c$ and $b \le c$, we have $j \le c$.

Similarly, the meet is an element m such that:

- first, $m \le a$ and $m \le b$, and
- second, for every element c for which $c \le a$ and $c \le b$, we have $c \le m$.

Historical comment. The idea to use lattices first appeared in [1]; see also [2,3].

Why lattices? A natural question is: why lattices? There are many partially ordered sets which are not lattice, so why namely lattices are mostly used?

What we do in this paper. In this paper, we provide a possible explanation of why lattices are used to describe degrees of confidence.

2. Analysis of the Problem

Let us start analyzing the problem: what are degrees of confidence? To resolve the mystery of using lattices, let us recall what are degrees of confidence, whet they are used for, and what are reasonable operations on these degrees.

We want to describe a set D of possible degrees of confidence. For some pairs of degrees a and b, we know that the degree b corresponds to the higher or same) confidence; we will denote this by $a \le b$. From this definition, it is clear that:

- for all a, we have $a \le a$, and
- if $a \le b$ and $b \le c$, i.e., if b corresponds to higher confidence than a and c corresponds to higher confidence than b, then c corresponds to higher confidence than a, i.e., $a \le c$.

These two properties mean that the relation $\le$ is a *partial order*.

Need for "and"- and "or"-operations. The expert's knowledge consists of several statements $S_1, \ldots, S_n$, for which of which we know the expert's degree of confidence $d_i = d(S_i)$ in this statement. Once we have elicited this knowledge from the expert(s), we can then use this knowledge to answer different queries Q.

In some cases, one of the available statements S_i already provides an answer to the query. In most cases, already, to answer the query, we need to combine several statements. For example, we can conclude that Q is true if we use two statements S_i and S_j. Since the experts are not 100% confidence in their statements, we are therefore not fully confident that "true" is the correct answer to this query. It is therefore desirable to provide the user not only with the "yes" answer, but also with the degree to which we are confident in this answer.

In the above case, our degree of confidence that the answer to the query Q is "true" is equal to the degree of confidence that the propositional combination $S_i \,\&\, S_j$ is true. In other case, we may have different propositional combinations.

We have collected degrees of confidence d_i corresponding to different statements d_i. It is known that the degree of confidence in $A \,\&\, B$ is not uniquely determined by our degrees of confidence in A and in B. For example, if we know nothing about A, then it is reasonable to say that $d(A) = d(\neg A) = 0.5$. In this case:

- for $B = A$, we have $A \,\&\, B \equiv A \,\&\, A \equiv A$ and thus, $d(A) = d(B) = 0.5$ and
$$d(A \,\&\, B) = d(A) = 0.5;$$
- on the other hand, for $B = \neg A$, the statement $A \,\&\, B \equiv A \,\&\, \neg A$ is clearly false, so we have $d(A) = d(B) = 0.5$ and
$$d(A \,\&\, B) = d(A) = 0.$$

So, ideally, we should not only elicit from the experts their degrees of belief d_i in different statements S_i, we should also elicit their degrees of belief in different propositional combinations of these statements. Unfortunately, this is not realistic: there are exponentially many propositional combinations, e.g., 2^n combinations of the type $S_1^{\varepsilon_1} \,\&\, \ldots \,\&\, S_n^{\varepsilon_n}$, where $\varepsilon_i \in \{-,+\}$, $S^+ \stackrel{\text{def}}{=} S$, and $S^- \stackrel{\text{def}}{=} \neg S$. For large $n \approx 300$, we have $2^{300} \approx 10^{90}$ – it is clearly not possible to ask that many questions to the expert.

Since we cannot elicit the expert's degree of belief in all possible propositional combinations, we thus need to be able to *estimate* these degrees of belief based on the expert's degree of belief in the basic statements.

In particular, we need to be able, given the degrees $a = d(A)$ and $b = d(B)$, to provide an estimate for the degree $d(A \,\&\, B)$. We will denote this estimate by $f_\&(a, b)$. The function $f_\&(a, b)$ that transforms the given values $a = d(A)$ and $b = d(B)$ into an estimate for $d(A \,\&\, B)$ is known as an *and*-operation.

Similarly, we need to be able, given the degrees $a = d(A)$ and $b = d(B)$, to provide an estimate for the degree $d(A \vee B)$. We will denote this estimate by $f_\vee(a,b)$. The function $f_\vee(a,b)$ that transforms the given values $a = d(A)$ and $b = d(B)$ into an estimate for $d(A \vee B)$ is known as an *or*-operation.

First reasonable property of "and"- and "or"-operations. Let us first consider the simplest case when conjunction $\&$ connects the statement S with itself, i.e., when we consider a propositional combination $S \,\&\, S$. Let $d = d(S)$ denote the expert's degree of confidence in the original statement S. In this case, when we apply an "and"-operation $f_\&(a,b)$ to estimate the expert's degree of confidence in $S \,\&\, S$, we get the estimate $f_\&(d,d)$.

For each statement S, the propositional combination $S \,\&\, S$ is simply equivalent to S. It is therefore reasonable to require that our estimate $f_\&(d,d)$ for the degree of confidence $d(S \,\&\, S)$ in the propositional combination $S \,\&\, S$ should coincide with the degree of confidence $d = d(S)$ in the original statement S, i.e., that we should have $f_\&(d,d) = d$ for all possible degrees d.

Similarly, since for each statement S, the propositional combination $S \vee S$ is equivalent to S, we should have $f_\vee(d,d) = d$ for all possible degrees d.

Second reasonable property of "and"- and "or"-operations. In general, the statement $A \,\&\, B$ is stronger than A and stronger than B. Thus, our degree of certainty in $A \,\&\, B$ cannot exceed the degree of certainty in A or B: $f_\&(a,b) \le a$ and $f_\&(a,b) \le b$.

Similarly, the statement $A \vee B$ is weaker than A and weaker than B. Thus, the degrees of certainty in A and in B cannot exceed our degree of certainty in $A \vee B$: $a \le f_\vee(a,b)$ and $b \le f_\vee(a,b)$.

Third reasonable property of "and"- and "or"-operations: monotonicity. Another reasonable property is monotonicity: if our degree of confidence in statements A and B increases, then the degree of confidence in propositional combinations $A \,\&\, B$ and $A \vee B$ should also increase – or at least remains the same. In other words, if $a \le a'$ and $b \le b'$, then we should have $f_\&(a,b) \le f_\&(a',b')$ and $f_\vee(a,b) \le f_\vee(a',b')$.

What we do. Let us show that these properties lead to the lattice structure.

3. Definitions and the Main Result

Definition 1. *By a* set of degrees*, we mean a partially ordered set $(D, \le)$ with two binary operations*

$$f_\& : D \times D \to D \text{ and } f_\vee : D \times D \to D$$

which satisfy the following three properties:

- *for each $d \in D$, we have*

$$f_\&(d,d) = f_\vee(d,d) = d;$$

- *for all $a, b \in D$, we have*

$$f_\&(a,b) \le a, \text{ and } f_\&(a,b) \le b,$$

$$a \le f_\vee(a,b), \text{ and } b \le f_\vee(a,b);$$

- *for all $a \le a'$ and $b \le b'$, we have*

$$f_\&(a,b) \le f_\&(a',b') \text{ and } f_\vee(a,b) \le f_\vee(a',b').$$

Proposition 1.

- *Every set of degrees is a lattice, with $f_\&(a,b)$ as meet and $f_\vee(a,b)$ as join.*
- *Every lattice is a set of degrees if we take meet as $f_\&(a,b)$ and join as $f_\vee(a,b)$.*

Proof. It is known that lattices satisfy all the properties which form our definition of a set of degrees. To prove our result, it is therefore sufficient to prove that each set of degrees is a lattice, with $f_\&(a,b)$ as meet and $f_\vee(a,b)$ as join.

1°. Let us first prove that for every two elements a and b, the value $f_\&(a,b)$ is a meet, i.e., that:

- the value $f_\&(a,b)$ is smaller than or equal to both a and b, i.e., that

$$f_\&(a,b) \le a \text{ and } f_\&(a,b) \le b,$$

  and
- the value $f_\&(a,b)$ is the largest of all the values c which are smaller than or equal to both a and b:

$$\text{if } c \le a \text{ and } c \le b, \text{ then } c \le f_\&(a,b).$$

Let us prove these two properties one by one.

1.1°. The first property, that $f_{\&}(a,b) \le a$ and $f_{\&}(a,b) \le b$, follows directly from the second property listed in the definition of a set of degrees.

1.2°. Let us now prove the second property, that if $c \le a$ and $c \le b$, then $c \le f_{\&}(a,b)$.

Indeed, due to the third property of a set of degrees, $c \le a$ and $c \le b$ imply that $f_{\&}(c,c) \le f_{\&}(a,b)$. By the first property of the set of degrees, $f_{\&}(c,c) = c$, so we indeed have $c \le f_{\&}(a,b)$.

2°. Let us now prove that for every two elements a and b, the value $f_\vee(a,b)$ is a join, i.e., that:

- the value $f_\vee(a,b)$ is greater than or equal to both a and b, i.e., that

$$a \le f_\vee(a,b) \text{ and } b \le f_\vee(a,b),$$

  and
- the value $f_\vee(a,b)$ is the smallest of all the values c which are larger than or equal to both a and b:

$$\text{if } a \le c \text{ and } b \le c, \text{ then } f_\vee(a,b) \le c.$$

Let us prove these two properties one by one.

2.1°. The first property, that $a \le f_\vee(a,b)$ and $b \le f_\vee(a,b)$, follows directly from the second property listed in the definition of a set of degrees.

2.2°. Let us now prove the second property, that if $a \le c$ and $b \le c$, then $f_\vee(a,b) \le c$. Indeed, due to the third property of a set of degrees, $a \le c$ and $b \le c$ imply that $f_\vee(a,b) \le f_\vee(c,c)$. By the first property of the set of degrees, we have $f_\vee(c,c) = c$, so we indeed have $f_\vee(a,b) \le c$.

The proposition is proven.

4. Auxiliary Result

A similar result can be proven for "and"- and "or"-operations with $n > 2$ inputs, that describe the degree of confidence in statements

$$S_1 \,\&\, \ldots \,\&\, S_n \text{ and } S_1 \vee \ldots \vee S_n :$$

Definition 2. *Let $n > 2$. By a* set of degrees with n-ary operations, *we mean a partially ordered set $(D, \le)$ with two n-ary operations*

$$f_{\&} : D^n \to D \text{ and } f_\vee : D^n \to D$$

which satisfy the following three properties:

- *for each $d \in D$, we have*

$$f_{\&}(d,\ldots,d) = f_\vee(d,\ldots,d) = d;$$

- *for all $a_1,\ldots,a_n \in D$, we have*

$$f_{\&}(a_1,\ldots,a_n) \le a_i \text{ for all } i, \text{ and}$$

$$a_i \le f_\vee(a_1,\ldots,a_n) \text{ for all } i;$$

- *for all $a_1 \le a_1'$, ..., $a_n \le a_n'$, we have*

$$f_{\&}(a_1,\ldots,a_n) \le f_{\&}(a_1',\ldots,a_n') \text{ and}$$

$$f_\vee(a_1,\ldots,a_n) \le f_\vee(a_1',\ldots,a_n').$$

Proposition 2. *For every set of degrees $(D, \le)$ with n-ary operations, and for every $a_1,\ldots,a_n \in D$:*

- *the value $f_{\&}(a_1,\ldots,a_n)$ is the meet (greatest lower bound) of the values $a_1,\ldots,a_n$, and*
- *the value $f_\vee(a_1,\ldots,a_n)$ is the join (least upper bound) of the values $a_1,\ldots,a_n$.*

Proof.

1°. Let us first prove that for every tuple $(a_1,\ldots,a_n)$, the value $f_{\&}(a_1,\ldots,a_n)$ is a meet, i.e., that:

- the value $f_{\&}(a_1,\ldots,a_n)$ is smaller than or equal to all the values $a_1,\ldots,a_n$: $f_{\&}(a_1,\ldots,a_n) \le a_i$ for all i, and
- the value $f_{\&}(a_1,\ldots,a_n)$ is the largest of all the values c which are smaller than or equal to all a_i:

$$\text{if } c \le a_1, \ldots, \text{ and } c \le a_n, \text{ then } c \le f_{\&}(a_1,\ldots,a_n).$$

Let us prove these two properties one by one.

1.1°. The first property, that $f_{\&}(a_1 \ldots, a_n) \le a_i$ for all i, follows directly from the second property listed in the definition of a set of degrees with n-ary operations.

1.2°. Let us now prove the second property, that if $c \le a_i$ for all i, then $c \le f_\&(a_1, \ldots, a_n)$.

Indeed, due to the third property of a set of degrees with n-art operations, $c \le a_1$, ..., and $c \le a_n$ imply that $f_\&(c, \ldots, c) \le f_\&(a_1, \ldots, a_n)$. By the first property of the set of degrees with n-ary operations, $f_\&(c, \ldots, c) = c$, so we indeed have $c \le f_\&(a_1, \ldots, a_n)$.

2°. Let us now prove that for every tuples $(a_1, \ldots, a_n)$, the value $f_\vee(a_1, \ldots, a_n)$ is a join, i.e., that:

- the value $f_\vee(a_1, \ldots, a_n)$ is greater than or equal to all the values a_i, i.e., that $a_i \le f_\vee(a_1, \ldots, a_n)$ for all i, and
- the value $f_\vee(a_1, \ldots, a_n)$ is the smallest of all the values c which are larger than or equal to all a_i:

  if $a_1 \le c$, ..., and $a_n \le c$, then
  $f_\vee(a_1, \ldots, a_n) \le c$.

Let us prove these two properties one by one.

2.1°. The first property, that $a_i \le f_\vee(a_1, \ldots, a_n)$ for all i, follows directly from the second property listed in the definition of a set of degrees with twi n-ary operations.

2.2°. Let us now prove the second property, that if $a_i \le c$ for all i, then $f_\vee(a_1, \ldots, a_n) \le c$. Indeed, due to the third property of a set of degrees with n-ary operations, $a_i \le c$ for all i implies that $f_\vee(a_1, \ldots, a_n) \le f_\vee(c, \ldots, c)$. By the first property of the set of degrees with n-ary operations, we have $f_\vee(c, \ldots, c) = c$, so we indeed have $f_\vee(a_1, \ldots, a_n) \le c$.

The proposition is proven.

5. Discussion

Main conclusion: we have a desired explanation of the use of lattices. The above result explains why lattices are a reasonable description of sets of degrees.

Additional conclusion: we not need to explicitly require commutativity or associativity. In our description of a set of degrees, we only used the fact that $A \,\&\, A$ means the same as A. There are other properties: e.g., since $A \,\&\, B$ means the same as $B \,\&\, A$, it is reasonable to require that the resulting estimates coincide, i.e., that $f_\&(a, b) = f_\&(b, a)$ – in mathematical terms, that the "and"-operation is commutative. Similarly, the fact that $A \,\&\, (B \,\&\, C)$ means the same as $(A \,\&\, B) \,\&\, C$ makes it reasonable to require that the "and"-operation is associative: $f_\&(a, f_\&(b, c)) = f_\&(f_\&(a, b), c)$. It also makes sense to similarly require that the "or"-operation be commutative and associative.

These requirements are part of the usual definitions of "and"-operations (t-norms) and "or"-operations (t-conorms) in fuzzy logic. Our proposition shows that it is not necessary to explicitly requires commutativity and associativity:

- even without these requirements, the above result implies that the set of degrees is a lattice, and
- in a lattice, meet and joint operations are always commutative and associative – e.g., the join is commutative by definition, since it is the smallest of all the values c which exceeds both a and b.

Lattices: pro and contra. In this paper, we showed that *if* we want to extend "and"- and "or"-operations to partially ordered sets, then it is reasonable to consider lattices. A natural next question is: *when* should we use general lattices – and when is it better to use the traditional $[0, 1]$-based fuzzy logic? The answer to this question is reasonably straightforward.

On the one hand, the more different degrees of confidence we have, the more adequately we can represent the subtleties of expert confidence. So, from the viewpoint of adequacy, lattices are desirable.

On the other hand, the more possible degrees we allow, the more time-consuming it is to elicit, store, and process all these degrees. So, from the viewpoint of practical applications – e.g., in intelligent control – we should use lattices if the gain of getting a control which more adequately describes the expert rules overcomes the loss of computation time needed to process the resulting degrees.

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