Log-Periodic Power Law as a Predictor of Catastrophic Events: A New Mathematical Justification

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Abstract

To decrease the damage caused by meteorological disasters, it is important to be able to predict these disasters as accurately as possible. One of the most promising ways of achieving such a prediction comes from the observation that in the vicinity of a catastrophic event, many parameters exhibit log-periodic power behavior, with oscillations of increasing frequency. By fitting the corresponding formula to the observations, it is often possible to predict the catastrophic event. Such successful predictions were made in many application areas ranging from ruptures of fuel tanks to earthquakes to stock market disruptions. The fact that similar formulas can be applied to vastly different systems seems to indicate that the log-periodic power behavior is not related to a specific nature of the system, it is caused by general properties of system. In this paper, we indeed provide a general system-based explanation of this law. The general character of this explanation makes us confident that this law can be also used to predict meteorological disasters.

1 Formulation of the Problem

Need to predict meteorological disasters. To decrease the damage caused by meteorological disasters, it is important to be able to predict these disasters as accurately as possible.

Let us try to use methods developed for predicting different types of disasters. A natural approach to solving the problem of predicting meteorological disasters it to see how similar disaster prediction problems are solved in other application areas.

In general, it is desirable to be able to predict all kinds of disaster, from mechanical disasters to catastrophic earthquakes to financial disasters.

One of the promising ways of achieving such a prediction comes from the observation that in the vicinity of a catastrophic event, many parameters exhibit so-called log-periodic power behavior, with oscillations of increasing frequency. Let us therefore describe this behavior in detail.

The emergence of log-periodic power law in disaster prediction. The history of log-periodic power law applications started with space exploration. To be able to safely return home, a spaceship needs to store fuel. This fuel needs to be protected. Such a protection is needed because in a space orbit, a satellite is moving at a speed of 8 km/sec, much faster than the speediest bullet. At such a speed, a micro-meteorite or a piece of space debris can easily penetrate a fuel tank, causing a catastrophic fuel leak. To avoid such a bullet-type penetration, engineers use the same material – Kevlar – that is used to prevent bullets fired by criminals from causing damage to police officers. The tests showed that while in general, Kevlar-coated tanks performed really well, on a few occasions, the Kevlar tanks catastrophically exploded.

By analyzing the telemetric records preceding these explosions, a physicist Didier Sornette noticed that an explosion is usually preceded by oscillations whose frequency increases as we approach the critical moment of time t_c. Moreover, he observed that the dependence of each corresponding parameter x on time t has the form

$$x(t) = A + B \cdot (t_c - t)^z + C \cdot (t_c - t)^z \cdot \cos(\omega \cdot \ln(t_c - t) + \varphi), \qquad (1)$$

for appropriate parameters A, B, C, z, ω, and φ. By fitting this model to the observations, he was able to show that, by monitoring such oscillations, we can predict the moment t_c of the catastrophic event reasonably well; see, e.g., [1, 13].

Sornette called the dependence (1) *Log-Periodic Power Law* (LPPL, for short).

Applications to earthquake prediction. Didier Sornette's wife, Anne Sauron-Sornette, is also a scientist: she is a geophysicist. Naturally, the two scientist spouses talk about their research. In general, mechanical ruptures are of great interest to geophysicists, since one of their main objectives is to study (and predict) earthquakes, and from the mechanical viewpoint, an earthquake is simply a mechanical rupture. Because of this analogy, they decided to check whether something similar to the log-periodic power law can be observed in earthquakes – and indeed, in many cases, they observed the log-periodic power law behavior in the period preceding an earthquake [7, 8, 12, 17, 19, 20].

Comment. It should be mentioned that this technique is *not* a panacea: not all earthquakes can be this predicted. However, some can be predicted, and the ability to predict an earthquake decreases the damage caused by this catastrophic event.

Financial applications. With some of his colleagues, Didier Sornette conjectured that many catastrophic financial events are similar to mechanical catastrophes – and indeed, they observed similar log-periodic fluctuations in the periods preceding crashes [24]. A similar observation was independently made in [4].

Both papers [4] and [24] appeared in 1996 in physics journals, and were not widely understood by economists. The situation changed drastically in 1997, when Didier Sornette and his colleague Olivier Ledoit, a management scientist, used their techniques in Summer 1997 to successfully predict the October 1997 market crash – and, by cleverly investing in put options, made a well-documented (and well-publicized) 400% profit on their investment.

This caused attention of economists, and now log-periodic power law predictions are important part of the econometric toolbox; see, e.g., [3, 5, 6, 9, 10, 11, 14, 15, 16, 21, 22, 23, 25, 26, 27].

Comment. Similarly to earthquakes, not all financial crashes can be thus predicted (see, e.g., [2]), but some crashes can be predicted, and the ability to predict at least some crashes can definitely decrease the financial risk.

Is there a general explanation for log-periodic power law? The fact that the same formula (1) can be applied to some different systems seems to indicate that the log-periodic power behavior is not related to the specific nature of the system, it is caused by general properties of system.

Some theoretical explanations have been provided in [11, 22], but these explanations are based on a very specific model of a system. It is desirable to come up with a more general explanation. If such a general explanation is found, this would make us confident that the corresponding law can also be used to predict meteorological disasters.

What we do in this paper. In this paper, we provide a general system-based explanation of this law. Thus, it is indeed reasonable to apply the log-periodic power law methodology to predict meteorological disasters.

2 Justification of the Log-Periodic Power Law: Motivations, Definitions, and the Main Result

A general description of a system's dynamics. We are interested in the dependence of quantities describing the system on time t: $x = x(t)$.

Need for finite-parametric families of functions. In principle, we can have arbitrary functions $x(t)$. However, our objective is to make predictions by using appropriate computer models. In the computer, at any given moment of time, we can only represent finitely many parameters. It is therefore reasonable to consider finite-parametric families of functions $x(t) = f(c_1, \ldots, c_n, t)$.

Taking into account that usually, we know an approximate model. Usually, we know the approximate dependence, i.e., we know the approximate values $c_1^{(0)}, \ldots, c_n^{(0)}$ of the corresponding parameters. In this case, the differences $\Delta c_i \stackrel{\text{def}}{=} c_i - c_i^{(0)}$ are small, so to find the values $c_i = c_i^{(0)} + \Delta c_i$ which fit with the observations, we can expand the dependence

$$x(t) = f(c_1, \ldots, c_n, t) = f(c_1^{(0)} + \Delta c_1, \ldots, c_n^{(0)} + \Delta c_n, t)$$

in Taylor series in terms of Δc_i and keep only linear terms in this expansion. As a result, we get a dependence of the type

$$x(t) = f_0(t) + \Delta c_1 \cdot e_1(t) + \ldots + \Delta c_n \cdot e_n(t),$$

where $f_0(t) \stackrel{\text{def}}{=} f(c_1^{(0)}, \ldots, c_n^{(0)}, t)$ and $e_i(t) \stackrel{\text{def}}{=} \dfrac{\partial f}{\partial c_i}$ for $i = 1, \ldots, n$.

Substituting the expressions $\Delta c_i = c_i - c_i^{(0)}$ into this formula, we conclude that

$$x(t) = e_0(t) + c_1 \cdot e_1(t) + \ldots + c_n \cdot e_n(t), \tag{2}$$

where we denoted

$$e_0(t) \stackrel{\text{def}}{=} f_0(t) - c_1^{(0)} \cdot e_1(t) - \ldots - c_n^{(0)} \cdot e_n(t).$$

In other words, the desired dependencies $x(t)$ are linear combinations of the appropriate functions $e_i(t)$.

How to select appropriate functions $e_i(t)$? To complete the description of the time dependence, we need to select the appropriate functions $e_i(t)$. To select these functions, we will use the symmetry ideas.

Natural symmetries: description. We want to describe a general model, a model that would be applicable to many different phenomena. The main objective of this model is to describe how different quantities depend on time t.

Let us take into account that the numerical value of time t depends on the choice of a starting point for measuring time and on the choice of the measuring unit.

For example, in most cases, we start our counting of time from the year 0 of our calendar system, but in many sports events, we start counting time with the moment this event started; the numerical results are quite different. In general, if we replace the starting point with a one which is s_0 units earlier, then the numerical value of time is increased by s_0, i.e., instead of the original numerical value t we get a new value $t' = t + s_0$. In mathematical terms, this transformation is known as a *shift*.

Similarly, we can measure time in years or in days or in seconds, the numerical results will be different. If we replace the original unit of time with a one which is λ times smaller, then the numerical value of time is multiplied by λ, i.e., instead of the original numerical value t we get a new value $t' = \lambda \cdot t$. For example, since a millisecond is $\lambda = 1000$ times smaller than a second, the original value of $t = 2.1$ seconds becomes $t' = 1000 \cdot 2.1 = 2100$ milliseconds. In mathematical terms, this transformation is known as *scaling*.

In general, if we use a different time unit and a different starting point, the numerical value of time changes from the original value t to the new value $t' = \lambda \cdot t + s_0$.

How to use the symmetries. We want to find the general functions $e_i(t)$, functions which would be applicable to all kinds of phenomena. It is therefore reasonable to require that the resulting class of functions does not change if we simply change the starting point or the measuring unit for measuring time. Thus, we arrive at the following definitions.

Definition 1. *By a* family of functions $\mathcal{F}$, *we mean a family consisting of all the functions of the type*

$$x(t) = e_0(t) + c_1 \cdot e_1(t) + \ldots + c_n \cdot e_n(t), \tag{2}$$

where the differentiable functions $e_0(t)$, $e_1(t)$, $\ldots$, $e_n(t)$ *are fixed, and the values* $c_1, \ldots, c_n$ *take arbitrary real values.*

Example. For example, if we take $e_0(t) = 0$, $e_1(t) = 1$, $e_2(t) = t$, $\ldots$, $e_i(t) = t^{i-1}$, $\ldots$, $e_n(t) = t^{n-1}$, then the corresponding family $\mathcal{F}$ consists of all the polynomials of degree $\le n-1$.

Definition 2. *We say that a family of functions* $\mathcal{F}$ *is* shift- and scale-invariant *if for every function* $x(t)$ *from this family, and for every two real numbers* λ *and* s_0, *the function* $y(t) \stackrel{\text{def}}{=} x(\lambda \cdot t + s_0)$ *also belongs to the family* $\mathcal{F}$.

Proposition 1. *Let* $\mathcal{F}$ *be a shift- and scale-invariant family of functions. Then all the functions from this family are polynomials.*

Comment. For readers' convenience, all the proofs are placed in the special Proofs section.

What happens when we have a catastrophic event. In the previous text, we considered situations of normal (smooth) evolution. In such a process, there is, in general, no special moment of time and no special time unit, so, in principle, we can select different starting moments and different time units.

What if at some moment t_c, we have a catastrophic event? In this case, we do have a fixed moment of time, so we can no longer arbitrarily select starting moment: a natural starting moment of time is t_c, so a natural way to describe time is as a different $T \stackrel{\text{def}}{=} t_c - t$. However, we can still select different time units, so we still have scaling transformation $T \to T' = \lambda \cdot T$, i.e., $t_c - t' = \lambda \cdot (t_c - t)$.

Definition 3. *We say that a family of functions* $\mathcal{F}$ *is* scale-invariant *if for every function* $x(T)$ *from this family, and for every real number* λ, *the function* $y(T) \stackrel{\text{def}}{=} x(\lambda \cdot T)$ *also belongs to the family* $\mathcal{F}$.

Proposition 2. *Let* $\mathcal{F}$ *be a scale-invariant family of functions. Then all the functions from this family are linear combinations of the functions of the type*

T^z, $T^z \cdot \cos(\omega \cdot \ln(T) + \varphi)$, and $T^z \cdot \cos(\omega \cdot \ln(T) + \varphi) \cdot (\ln(T))^k$ for real values z, ω, φ and a natural number $k \geq 0$.

Comment. Since $T = t_c - t$, in terms of the original time t, we get a linear combination of functions of the type $(t_c - t)^z$, $(t_c - t)^z \cdot \cos(\omega \cdot \ln(t_c - t)) + \varphi)$, and $(t_c - t)^z \cdot \cos(\omega \cdot \ln(t_c - t) + \varphi) \cdot (\ln(t_c - t))^k$. Thus, we indeed explain the semi-empirical formula (1).

3 Auxiliary Result: Log-Periodic Power Functions Are Optimal (in Some Reasonable Sense)

What we proved so far. In the previous section, we proved that if we require that a class of approximating functions is invariant, then the corresponding class consists of log-periodic power functions.

Optimization: a more natural way of selecting approximating functions. While invariance makes sense, a more natural way of selecting a family of approximating functions is to select a family which is optimal – in some reasonable sense.

In this section, we show that log-periodic power functions are not only invariant, they are also optimal – in some reasonable sense.

What is an optimality criterion? When we say "optimal", we mean that on the set of all such families, there must be a relation $\succeq$ describing which family is better or equal in quality.

This relation must be transitive: if $\mathcal{F}$ is better than $\mathcal{F}'$, and $\mathcal{F}'$ is better than $\mathcal{F}''$, then $\mathcal{F}$ is better than $\mathcal{F}''$.

This relation is not necessarily asymmetric, because we can have two approximating families of the same quality. However, we would like to require that this relation be *final* in the sense that it should define the unique *optimal* family $\mathcal{F}_{\text{opt}}$, i.e., the unique family for which $\forall \mathcal{F}\, (\mathcal{F}_{\text{opt}} \succeq \mathcal{F})$. Indeed:

- If none of the families is optimal, then this criterion is of no use, so there should be *at least one* optimal family.
- If *several* different families are equally optimal, then we can use this ambiguity to optimize something else: e.g., if we have two families with the same approximating quality, then we choose the one which is easier to compute. As a result, the original criterion was not final: we get a new criterion ($\mathcal{F} \succeq_{\text{new}} \mathcal{F}'$ if either $\mathcal{F}$ gives a better approximation, or if $\mathcal{F} \sim_{\text{old}} \mathcal{F}'$ and $\mathcal{F}$ is easier to compute), for which the class of optimal families is narrower. We can repeat this procedure until we get a final criterion for which there is only one optimal family.

An optimality criterion should be invariant. It is reasonable to require that what is better in one representation should be better in another representation as well. In other words, it is reasonable to require the relation $\mathcal{F} \succeq \mathcal{F}'$ should be invariant relative to the re-scaling $T \to T' = \lambda \cdot T$.

Definition 4. *For every family of functions $\mathcal{F}$ and for every λ, by a λ-*rescaling *of $\mathcal{F}$, we mean the family of all the functions $x(\lambda \cdot T)$ for all $x(T) \in \mathcal{F}$. This re-scaling will be denoted by $S_\lambda(\mathcal{F})$.*

Comment. One can easily check that if $\mathcal{F}$ is a family of family of functions in the sense of Definition 1, then $S_\lambda(\mathcal{F})$ is also a family of functions. Indeed, if $\mathcal{F}$ consists of all the functions of the type

$$e_0(T) + c_1 \cdot e_1(T) + \ldots + c_n \cdot e_n(T),$$

then $S_\lambda(\mathcal{F})$ consists of all the functions of the type

$$e'_0(T) + c_1 \cdot e'_1(T) + \ldots + c_n \cdot e'_n(T),$$

where $e'_i(T) \stackrel{\text{def}}{=} e_i(\lambda \cdot T)$.

Definition 5. *Let $\mathcal{C}$ be a class of all families of functions.*

- *By an* optimality criterion*, we mean a* pre-ordering *(i.e., a transitive reflexive relation) $\preceq$ on the class $\mathcal{C}$.*
- *We say that an optimality criterion is* scale-invariant *if for all λ, and for all $\mathcal{F}, \mathcal{F}' \in \mathcal{C}$, $\mathcal{F} \preceq \mathcal{F}'$ implies $S_\lambda(\mathcal{F}) \preceq S_\lambda(\mathcal{F}')$.*
- *We say that an optimality criterion is* final *if there exists one and only one element $\mathcal{F}_{\text{opt}} \in \mathcal{C}$ that is preferable to all the others, i.e., for which $\mathcal{F} \preceq \mathcal{F}_{\text{opt}}$ for all $\mathcal{F} \neq \mathcal{F}_{\text{opt}}$.*

Proposition 3. *Let $\preceq$ be a scale-invariant and final optimality criterion on the class $\mathcal{C}$ of all families of functions. Then, every function from the optimal family $\mathcal{F}_{\text{opt}}$ is a linear combination of the functions of the type T^z, $T^z \cdot \cos(\omega \cdot \ln(T) + \varphi)$, and $T^z \cdot \cos(\omega \cdot \ln(T) + \varphi) \cdot (\ln(T))^k$ for real values z, ω, φ and a natural number $k \geq 0$.*

4 Proofs

Comment. The main ideas of the proofs are similar to the proofs from [18].

Proof of Proposition 2. Let us start by proving Proposition 2.

1°. Scale-invariance means that if we have a function $x(T)$ from the class $\mathcal{F}$, then, for every λ, the function $x(\lambda \cdot T)$ also belongs to this class.

Each function $x(T)$ from the class $\mathcal{F}$ can be represented as a linear combination $\sum\limits_{i=0}^{n} k_i \cdot a_i(T)$ of the functions $a_i(T) \in \mathcal{F}$, where $a_0(T) = e_0(T)$,

$a_i(T) = e_0(T) + e_i(T)$ for $i \geq 1$, and $\sum\limits_{i=0}^{n} k_i = 1$. Thus, it is sufficient to prove that $a_i(\lambda \cdot T) \in \mathcal{F}$, i.e., that we have

$$a_i(\lambda \cdot T) = k_{i0}(\lambda) \cdot a_0(T) + k_{i1}(\lambda) \cdot a_1(T) + \ldots + k_{in}(\lambda) \cdot a_n(T) \tag{3}$$

for appropriate values $k_{ij}(\lambda)$ depending on λ.

2°. Since the functions $e_i(T)$ are differentiable, the functions $a_i(T)$ are differentiable as well. For each i, if we select $n+1$ different values $T_0, \ldots, T_n$, then for $n+1$ unknowns $k_{i0}(\lambda)$, $\ldots$, $k_{in}(\lambda)$, we get a system of $n+1$ linear equations

$$\begin{gathered} a_i(\lambda \cdot T_0) = k_{i0}(\lambda) \cdot a_0(T_0) + k_{i1}(\lambda) \cdot a_1(T_0) + \ldots + k_{in}(\lambda) \cdot a_n(T_0); \\ \ldots \\ a_i(\lambda \cdot T_n) = k_{i0}(\lambda) \cdot a_0(T_n) + k_{i1}(\lambda) \cdot a_1(T_n) + \ldots + k_{in}(\lambda) \cdot a_n(T_n). \end{gathered} \tag{4}$$

By using the Cramer's rule, we can describe the solutions $k_{ij}(\lambda)$ of this system of equations as a differentiable function in terms of $e_i(\lambda \cdot T_k)$ and $e_i(T_k)$. Since the functions $a_i(T)$ are differentiable, we conclude that the functions $k_{ij}(\lambda)$ are differentiable as well.

3°. Now that we know that all the functions in the equation (3) are differentiable, we can differentiate both sides of each equation (3) with respect to λ. As a result, we get the following equation:

$$T \cdot \dot{a}_i(\lambda \cdot T) = \dot{k}_{i0}(\lambda) \cdot a_0(T) + \ldots + \dot{k}_{in}(\lambda) \cdot a_n(T), \tag{5}$$

where $\dot{g}$ denotes the derivative of the function g. In particular, for $\lambda = 1$, we get

$$T \cdot \frac{da_i}{dT} = C_{i0} \cdot a_0(T) + \ldots + C_{in} \cdot a_n(T), \tag{6}$$

where we denoted $C_{ij} \stackrel{\text{def}}{=} \dot{k}_{ij}(1)$. This system of differential equations can be further simplified if we take into account that $\dfrac{dT}{T} = dS$, where $S \stackrel{\text{def}}{=} \ln(T)$. Thus, if we take a new variable $S = \ln(T)$ for which $T = \exp(S)$ and new unknowns $A_i(S) \stackrel{\text{def}}{=} a_i(\exp(S))$, the above equations take a simplified form

$$\frac{dA_i}{dS} = C_{i0} \cdot A_0(S) + \ldots + C_{in} \cdot A_n(S). \tag{7}$$

Equations (7) corresponding to $i = 0, 1, \ldots, n$ form a system of linear differential equations with constant coefficients.

4°. A general solution to a system of linear differential equations with constant coefficients is well known; thus, each function $A_i(S)$ is a linear combination of functions of the type $\exp(z \cdot S)$, $S^k \cdot \exp(z \cdot S)$, $\exp(z \cdot S) \cdot \cos(\omega \cdot S + \varphi)$, and $S^k \cdot \exp(z \cdot S) \cdot \cos(\omega \cdot S + \varphi)$, for some real numbers z, ω, and φ, and for a natural number $k \geq 1$.

To represent these expressions in terms of T, we need to substitute $S = \ln(T)$ into the above formulas. Here,

$$\exp(z \cdot S) = \exp(z \cdot \ln(T)) = (\exp(\ln(T))^z = T^z.$$

Thus, we conclude that each function $a_i(T)$ is a linear combination of functions of the form T^z, $T^z \cdot (\ln(T))^k$, $T^z \cdot \cos(\omega \cdot \ln(T) + \varphi)$, and

$$T^z \cdot \cos(\omega \cdot \ln(T) + \varphi) \cdot (\ln(T))^k.$$

Since every function $x(T)$ from the family $\mathcal{F}$ is a linear combination of the functions $a_i(T)$, the function $x(T)$ is also a linear combination of the above functions. The proposition is proven.

Proof of Proposition 1. In this case, in addition to scale-invariance, the class of functions $\mathcal{F}$ is also shift-invariant.

1°. Shift-invariance means that if we have a function $x(t)$ from the class $\mathcal{F}$, then, for every real number s_0, the function $x(t + s_0)$ also belongs to this class.

We have already mentioned, in the proof of Proposition 2, that each function $x(t)$ from the class $\mathcal{F}$ is a linear combinations of the functions $a_i(t)$, where $a_0(t) = e_0(t)$ and $a_i(t) = e_o(t) + e_i(t)$ for $i \geq 1$. It is therefore sufficient to require that this property be satisfied for the functions $a_i(t)$, i.e., that we have

$$a_i(t + s_0) = s_{i0}(s_0) \cdot a_0(t) + s_{i1}(s_0) \cdot a_1(t) + \ldots + s_{in}(s_0) \cdot a_n(t) \qquad (8)$$

for appropriate values $s_{ij}(s_0)$ depending on s_0.

2°. Similarly to the proof of Proposition 2, the functions $a_i(t)$ are differentiable. For each i, if we select $n+1$ different values $t_0, \ldots, t_n$, then for $n+1$ unknowns $s_{i0}(s_0)$, $\ldots$, $s_{in}(s_0)$, we get a system of $n+1$ linear equations

$$a_i(t_0 + s_0) = s_{i0}(s_0) \cdot a_0(t_0) + s_{i1}(s_0) \cdot a_1(t_0) + \ldots + s_{in}(s_0) \cdot a_n(t_0);$$
$$\ldots \qquad (9)$$
$$a_i(t_n + s_0) = s_{i0}(s_0) \cdot a_0(t_n) + s_{i1}(s_0) \cdot a_1(t_n) + \ldots + s_{in}(s_0) \cdot a_n(t_n).$$

By using the Cramer's rule, we can describe the solution $s_{ij}(s_0)$ of this system of equations as a differentiable function in terms of $a_i(t_k + s_0)$ and $a_i(t_k)$. Since the functions $a_i(t)$ are differentiable, we conclude that the functions $s_{ij}(s_0)$ are differentiable as well.

3°. Now that we know that all the functions in the equation (8) are differentiable, we can differentiate both sides of each equation (8) with respect to s_0. As a result, we get the following equation:

$$\dot{a}_i(t + s_0) = \dot{s}_{i0}(s_0) \cdot a_0(t) + \ldots + \dot{s}_{in}(s_0) \cdot a_n(t). \qquad (10)$$

In particular, for $s_0 = 0$, we get

$$\frac{da_i}{dt} = S_{i0} \cdot a_0(t) + \ldots + S_{in} \cdot a_n(t), \qquad (11)$$

where we denoted $S_{ij} \stackrel{\text{def}}{=} \dot{s}_{ij}(0)$. Equations (11) corresponding to $i = 0, 1, \ldots, n$ form a system of linear differential equations with constant coefficients.

4°. As we have mentioned in the proof of Proposition 2, a general solution to a system of linear differential equations with constant coefficients is well known: it is a linear combination of functions of the type $\exp(z \cdot t)$, $t^k \cdot \exp(z \cdot t)$, $\exp(z \cdot t) \cdot \cos(\omega \cdot t + \varphi)$, and $t^k \cdot \exp(z \cdot t) \cdot \cos(\omega \cdot t + \varphi)$, for some real numbers z, ω, and φ, and for a natural number $k \geq 1$. Thus, each function $a_i(t)$ is a linear combination of these functions.

Since the family $\mathcal{F}$ is also scale-invariant, according to Proposition 2, each function $a_i(t)$ must also be a linear combination of the functions

$$t^z, \quad t^z \cdot (\ln(t))^k, \quad t^z \cdot \cos(\omega \cdot \ln(t) + \varphi), \text{ and } t^z \cdot \cos(\omega \cdot \ln(t) + \varphi) \cdot (\ln(t))^k.$$

One can check that the only functions which can be described as linear combinations of functions from both lists are linear combinations of functions of the type t^k for some natural number $k \geq 0$.

Thus, each function $a_i(t)$ is a linear combination of monomials t^k and is, thus, a polynomial. Every function $x(t)$ from the family $\mathcal{F}$ is a linear combination of the functions $a_i(t)$ and is, thus, also a polynomial. The proposition is proven.

Proof of Proposition 3. Since the criterion $\preceq$ is final, there exists one and only one optimal family of sets. Let us denote this family by $\mathcal{F}_{\text{opt}}$.

Let us first show that this family $\mathcal{F}_{\text{opt}}$ is scale-invariant, in the sense that $S_\lambda(\mathcal{F}_{\text{opt}}) = \mathcal{F}_{\text{opt}}$ for every λ.

Indeed, let λ be any non-zero real number. From the optimality of $\mathcal{F}_{\text{opt}}$, we conclude that for every $\mathcal{F} \in \mathcal{C}$, we have $S_{1/\lambda}(\mathcal{F}) \preceq \mathcal{F}_{\text{opt}}$. From the scale-invariance of the optimality criterion, and from the fact that $S_\lambda(S_{1/\lambda}(\mathcal{F})) = \mathcal{F}$, we can now conclude that $\mathcal{F} \preceq S_\lambda(\mathcal{F}_{\text{opt}})$. This is true for all $\mathcal{F} \in \mathcal{C}$ and therefore, the family $S_\lambda(\mathcal{F}_{\text{opt}})$ is optimal. But since the optimality criterion is final, there is only one optimal family; hence, $S_\lambda(\mathcal{F}_{\text{opt}}) = \mathcal{F}_{\text{opt}}$. So, $\mathcal{F}_{\text{opt}}$ is indeed invariant.

Now, the result follows from Proposition 2.

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