Asymptotically Optimal Algorithm for Checking Whether a Given Vector Is a Solution to a Given Interval-Quantifier Linear System*

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Abstract

In many practical situations, we have a linear dependence between different quantities. In such situations, we often need to solve the corresponding systems of linear equations. Often, we know the parameters of these equations with interval uncertainty. In this case, depending on the practical problem, we have different notions of a solution. For example, if we determine parameters from observations, we are interested in all the unknowns x_j which satisfy the given system of linear equations for some possible values of the parameters. If we design a system so that it does not exceed given tolerance bounds, then we need to make sure that for all possible values of the design parameters there exist possible values of the outcome parameters for which the system is satisfied, etc. In general, we can have an arbitrary sequence of quantifiers corresponding to different parameters. The resulting systems are known as interval-quantifier linear systems.

In this paper, we provide an asymptotically optimal algorithm for checking whether a given vector is a solution to a given interval-quantifier linear system. For a system of m equations with n unknown, this algorithm takes time $O(m \cdot n)$.

1 Formulation of the Problem

Dependencies are important for science and engineering. One of the main objectives of science is to be able to predict the future state of the world. A state of the world is described by the values of different physical quantities, so we can say that one of the main objectives of science is to predict the values of different physical

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quantities. We want to predict tomorrow's temperature, we want to predict the future position of a spaceship, we want to predict how a patient will react to a certain drug.

Once we know how to predict the results of different actions, a natural next step is to come up with actions and/or decisions that would lead to the desired result: how to stop global warming, how to make sure that the spaceship reaches its destination, how to select the medicine that will be the best for the given patient.

In all these situations, the fact that we can use the values of some quantities at some moments of time to predict the values of the same (or other) quantities at different moments of time means that there is a dependence between different quantities. Once we know the function $q = f(p_1, \ldots, p_n)$ describing how the desired quantity q depends on the known quantities $p_1, \ldots, p_n$, we can use this function and the known values p_j to predict the value q.

Once we know the dependence $q_i = f(p_1, \ldots, p_n)$ between desired difficult-to-change quantities q_i and easier-to-change quantities p_j, then, to achieve the desired values q_i, we can find the corresponding values p_i by solving the corresponding system of equations.

Dependencies are often linear.   In many cases, the range of possible values of the quantities p_i is small, i.e., all possible values p_i are close to some "nominal" value $p_i^{(0)}$. In such cases, we can expand the dependence $q = f(p_1, \ldots, p_n)$ in Taylor series around the nominal point $p^{(0)} = (p_1^{(0)}, \ldots, p_n^{(0)})$, and ignore terms which are quadratic (or higher order) in terms of the small difference $\Delta p_j \stackrel{\text{def}}{=} p_j - p_j^{(0)}$:

$$q = c_0 + \sum_{j=1}^{n} c_j \cdot \Delta p_j.$$

Substituting $\Delta p_j = p_j - p_j^{(0)}$ into this formula, we conclude that q linearly depends on p_j:

$$q = q_0 + \sum_{j=1}^{n} c_j \cdot p_j. \tag{1}$$

Need to solve systems of linear equations.   As we have mentioned, in many practical situations, we need to solve systems of equations. When the dependence is linear, we need to solve systems of linear equations. Let us give two simple examples of such need.

To be able to use linear dependence (1), we need to know the values of the coefficients q_0 and c_j. These coefficients can be determined if we repeatedly measure the values $p_1, \ldots, p_n$ and the corresponding value q. If we denote, by $p_{k1}, \ldots, p_{kn}$, and q_k, the values corresponding to the k-th measurement, then we get a system of linear equations

$$q_k = q_0 + \sum_{j=1}^{n} c_i \cdot p_{ki} \tag{2}$$

with $n + 1$ unknowns $q_0, c_1, \ldots, c_n$.

Once we know the coefficients $q_{i0}, c_{i1}, \ldots, c_{in}$ describing the dependence of different desired quantities q_i on the quantities p_j:

$$q_i = q_{i0} + \sum_{j=1}^{n} c_{ij} \cdot p_j, \tag{3}$$

we can then, for every desired tuple of values $q_1, q_2, \ldots$, solve the system of linear equations (3) and find the values p_j which guaranteed the desired values of the quantities p_i.

In all these cases, we need to consider a system of linear equations

$$\sum_{j=1}^{n} a_{ij} \cdot x_j = b_i, \quad i = 1, \ldots, m, \tag{4}$$

in which we know the coefficients a_{ij} and b_i, and we need to find the values x_j.

Checking whether a given vector satisfies the given system of linear equations: computational complexity. When we solve a system of equations, it is often a good idea to check whether the result of a system-solving algorithm actually solves the original system. For a linear system, such a checking is easy: we plug in the checked tuple $x = (x_1, \ldots, x_n)$ into each of the m equations (4), and we check whether the sum $\sum_{j=1}^{n} a_{ij} \cdot x_j$ is equal to b_i.

For each equation i, we perform n multiplications to compute the products $a_{ij} \cdot x_j$, and then $n - 1$ additions to add the products. For all m equations, this algorithm takes $m \cdot (n + (n - 1)) = O(n \cdot m)$ computation steps.

One can easily show that this algorithm is asymptotically optimal, in the sense that it is not possible to have a much faster algorithm. Indeed, any algorithm must process the values of all $n \cdot m + m + n$ quantities a_{ij}, b_i, and x_j (at least when all of them are non-zeros): if one of these values is not used and the algorithm concludes that the tuples satisfies the system, then this algorithm would return the same conclusion for any other value of the unused quantity – but by changing this value, we can always make at least one of the equations false. Each elementary operation takes at most two inputs, so to process $\geq n \cdot m$ numbers, we must use at least $(n \cdot m)/2$ operations.

Thus, the above simple algorithm takes time $\leq C \cdot n \cdot m$, while any algorithm requires time $\geq c \cdot n \cdot m$. In other words, modulo a multiplicative constant, the above simple algorithm is optimal.

What if we have a system of equalities and inequalities. In some cases, we want to maintain a certain value of a quantity: for example, an ideal air conditioner should maintain the most comfortable temperature. In other cases, the requirements are relaxed: for example, when we store meat in the freezer, it is OK if the temperature change a little bit as long as stays blow freezing. In such cases, instead of an equation, we have an inequality. In general, we have a system consisting of equations and inequalities

$$\left(\sum_{j=1}^{n} a_{ij} \cdot x_j \right) \sigma_i \, b_i, \quad i = 1, \ldots, m, \tag{5}$$

where $\sigma_i \in \{=, \geq, \leq\}$.

For such situations, we can similarly directly check, in time $O(n \cdot m)$, whether a given vector x satisfies this system of equalities and inequalities, and this algorithm is also asymptotically optimal.

Need for interval uncertainty. In practice, measurements are never exact; a measurement result $\widetilde{p}$, in general, different from the actual (unknown) value p of the corresponding quantity. In many cases, the only information that we have about the

measurement error $\Delta p \overset{\text{def}}{=} \widetilde{p} - p$ is the upper bound Δ on its absolute value provided by the manufacturer of the measuring instrument: $|\Delta p| \leq \Delta$; see, e.g., [4]. In this situation, after the measurement, the only information that we have about the actual value p is that this value belongs to the interval $\boldsymbol{p} = [\widetilde{p} - \Delta, \widetilde{p} + \Delta]$.

In particular, for linear systems, we have a system (5), in which the only available information consists of the intervals $\boldsymbol{a}_{ij}$ that contain the actual (unknown) values a_{ij} and the intervals $\boldsymbol{b}_i$ that contain the actual (unknown) values b_i.

Need for interval-quantifier systems.   When the values a_{ij} and b_i are known exactly, it is clear what it means for a vector x to solve this system: all the equalities and inequalities must be satisfied.

Under interval uncertainty, the notion of a solution depends on the practical problem. For example, if we want to find the coefficients x_i of the linear dependence from the observations, then we need to consider all possible tuples x_i corresponding to different combination of values $a_{ij} \in \boldsymbol{a}_{ij}$ and $b_i \in \boldsymbol{b}_i$. Different values a_{ij} and b_i from these intervals lead, in general, to different tuples x. In this case, a vector x satisfies the system (5) if it satisfies this system for *some* possible values of a_{ij} and b_i, i.e., if

$$\exists a_{11} \in \boldsymbol{a}_{11} \, \exists a_{12} \in \boldsymbol{a}_{12} \, \ldots \, \exists b_n \in \boldsymbol{b}_n \, \forall i \left(\left(\sum_{j=1}^{n} a_{ij} \cdot x_j \right) \sigma_i \, b_i \right). \tag{6}$$

The set of all such solutions is known as the *united solution* to the corresponding interval linear system.

On the other hand, in the case of all equalities, for the decision problem, if we want to make sure that the resulting values b_i are within the given intervals $\boldsymbol{b}_i$ for all possible values $a_{ij} \in \boldsymbol{a}_{ij}$, the problem takes the form

$$\forall a_{11} \in \boldsymbol{a}_{11} \, \ldots \, \forall a_{mn} \in \boldsymbol{a}_{mn} \, \exists b_1 \in \boldsymbol{b}_1 \, \ldots \, \exists b_n \in \boldsymbol{b}_n \, \forall i \left(\sum_{j=1}^{n} a_{ij} \cdot x_j = b_i \right). \tag{7}$$

The set of all such solutions is known as the *tolerance solution* to the corresponding interval linear system.

There are many other possible problems. In general, we can have a system (5) preceded by an arbitrary sequence of quantifiers $\exists a_{ij} \in \boldsymbol{a}_{ij}$, $\forall a_{ij} \in \boldsymbol{a}_{ij}$, $\exists b_i \in \boldsymbol{b}_i$, and $\forall b_i \in \boldsymbol{b}_i$, one for each of the variables a_{ij} and b_i. Such a system is called a *interval-quantifier linear system*.

Natural question: how complex is it to check whether a give vector x is a solution to the given interval-quantifier system?   A natural question is: how to check that a given vector is a solution to a given system? Because of the quantifiers, we can no longer check this directly: a direct checking would mean that we check infinitely many values $a_{ij} \in \boldsymbol{a}_{ij}$.

A related negative result.   Instead of checking whether a given vector is a solution, we may want to describe the set of all solutions – e.g., for each j, we can try to describe the interval of possible values of x_j corresponding to all possible solutions $x = (x_1, \ldots, x_n)$. It turns out that computing such an interval is NP-hard; see, e.g., [1, 2].

Known positive results.   For united solutions to a system of interval linear equations, Oettli and Prager [3] showed that a vector x is a solution if and only if the following set of inequalities is satisfied:

$$\left| \sum_{j=1}^{n} \widetilde{a}_{ij} - \widetilde{b}_i \right| \leq \Delta_{bi} + \sum_{j=1}^{n} \Delta_{aij} \cdot |x_j|,$$

where $\widetilde{a}_{ij}$ and $\widetilde{b}_i$ are midpoints of the corresponding intervals, and Δ_{aij} and Δ_{bi} are the half-widths of these intervals. These inequalities can be checked in time $O(n \cdot m)$; thus, in this case, we also have an asymptotically optimal checking algorithm.

In [6], this algorithm was extended to a more general AE case of interval-quantifier equalities, i.e., to the case when we have a sequence of universal quantifiers followed by a sequence of existential quantifiers.

The paper [5] showed that when all relations are inequalities, a similar algorithm can be designed for all possible combinations of quantifiers.

What we do in this paper.   In this paper, we show that an asymptotically optimal $O(n \cdot m)$ algorithm is possible for all possible sequences of quantifiers and for all possible combination of equalities and inequalities.

2   Main Result

Definition 1.   [5] *By an* interval-quantifier linear system*, we mean a formula F which is obtained from the formula*

$$\bigwedge_{i=1}^{m} \left(\left(\sum_{j=1}^{n} a_{ij} \cdot x_j \right) \sigma_i \, b_i \right)$$

where $\sigma_i \in \{=, \geq, \leq\}$*, by applying quantifiers* $\exists a_{ij} \in \boldsymbol{a}_{ij}$*,* $\forall a_{ij} \in \boldsymbol{a}_{ij}$*,* $\exists b_i \in \boldsymbol{b}_i$*, and* $\forall b_i \in \boldsymbol{b}_i$*, where* $\boldsymbol{a}_{ij}$ *and* $\boldsymbol{b}_i$ *are intervals with rational endpoints.*

Definition 2.   [5] *We say that a given rational-valued tuple* $x = (x_1, \ldots, x_n)$ *satisfies (or a solution to) the interval-quantifier linear system if the corresponding formula F is true for this* x*.*

Proposition.

- *Every algorithm for checking whether a given tuple satisfies a given interval-quantifier linear system needs at least* $c \cdot m \cdot n$ *computation steps.*

- *There exists an algorithm that checks, in time* $O(m \cdot n)$*, whether a given tuple* x *satisfies a given interval-quantifier linear system.*

Comment.   The first part of the proposition implies that the algorithm mentioned in the second part is asymptotically optimal.

Proof.

1°. Let us first prove that we cannot check whether x is a solution in time $< (m \cdot n)/2$. Moreover, we will prove that such "fast" checking is impossible even for the case when all the values x_j are different from 0, and we have a system of linear equations with exactly known values a_{ij} and b_i, i.e., when all intervals are degenerate and all relations σ_i are equalities.

Indeed, if the tuple satisfies the system, then we need to process all $m \cdot n$ values a_{ij} – because if we do not use one of the values, the algorithm will return the same answer "satisfies" for all real values $a'_{ij} \neq a_{ij}$, and this is not possible, since for $a'_{ij} \neq a_{ij}$, the i-th equation $\sum_{j=1}^{n} a_{ij} \cdot x_j = b_i$ will no longer be satisfied. Processing a value means that there should be at least one elementary operation involving these values. Each elementary operation involves at most 2 values, so to process all $m \cdot n$ parameters a_{ij}, we need at least $(m \cdot n)/2$ elementary operations.

The first part of the proposition is proven.

2°. Let us now construct an algorithm that checks whether x is a solution in time $O(m \cdot n)$. This algorithm is, in effect, a minor modification of an algorithm described in [5].

Before we do anything, let us dismiss the terms x_j for which $x_j = 0$, since for these terms, the product $a_{ij} \cdot x_j$ is always equal to 0 and thus, does not contribute anything to the corresponding sums. Thus, without losing generality, we can assume that $x_j \neq 0$ for all j.

2.1°. First, we can take into account that $a = b$ is equivalent to $a - b = 0$, $a \geq b$ is equivalent to $a - b \geq 0$, and $a \leq b$ is equivalent to $a - b \leq 0$. Thus, $a \, \sigma \, b$ is equivalent to $(a - b) \, \sigma \, 0$. In particular, each condition $\left(\sum_{j=1}^{n} a_{ij} \cdot x_j \right) \sigma_i \, b_i$ is equivalent to $\left(\sum_{j=1}^{n} a_{ij} \cdot x_j - b_i \right) \sigma_i \, 0$.

This condition can be simplified if we take $x_0 \stackrel{\text{def}}{=} -1$ and define $a_{i0} \stackrel{\text{def}}{=} b_i$. In these terms, the condition $\left(\sum_{j=1}^{n} a_{ij} \cdot x_j - b_i \right) \sigma_i \, 0$ takes the form $\left(\sum_{j=0}^{n} a_{ij} \cdot x_j \right) \sigma_i \, 0$.

For all three possible relations σ_i, this condition can be represented in the form

$$\underline{c}_i \leq \sum_{j=0}^{n} a_{ij} \cdot x_j \leq \overline{c}_i : \tag{8}$$

- when σ_i is $=$, we take $\underline{c}_i = \overline{c}_i = 0$;
- when σ_i is $\geq$, we take $\underline{c}_i = 0$ and $\overline{c}_i = +\infty$;
- when σ_i is $\leq$, we take $\underline{c}_i = -\infty$ and $\overline{c}_i = 0$.

When we represent b_i as a_{i0}, then all quantifiers take the form $\exists a_{ij} \in \boldsymbol{a}_{ij}$ or $\forall a_{ij} \in \boldsymbol{a}_{ij}$.

2.2°. We will process the given system quantifier-by-quantifier, starting with the inmost quantifiers. We start with the system (8). In the beginning, none of the quantifiers have been processed, so the set I of the pairs (i, j) corresponding to not-yet-processed quantifiers consists of all possible pairs of values $i = 1, \ldots, m$ and $j =$

$0, \ldots, n$. We will show that after we process each quantifier (i, j), we either conclude that x is not a solution to the original system, or we get a similar system with remaining quantifiers in front of the relations

$$\underline{c}_i \leq \sum_{j:(i,j) \in I} a_{ij} \cdot x_j \leq \overline{c}_i, \tag{9}$$

for appropriately changed rational values $\underline{c}_i$ and $\overline{c}_i$ corresponding to the processed quantifier (and all other inequalities will be unchanged);

As a result, once we eliminate all $m \cdot (n + 1)$ quantifiers, we will get $I = \emptyset$ and thus, the system (9) will reduce to a system easy-to-check inequalities $\underline{c}_i \leq 0 \leq \overline{c}_i$ between rational numbers.

2.3°. Let us first consider the case when the quantifier has the form $\exists a_{ij} \in [\underline{a}_{ij}, \overline{a}_{ij}]$. If we apply this quantifier to the set of conditions (9), then, since this quantifier affects only the i-th condition (9), we get the condition

$$\exists a_{ij} \left((\underline{a}_{ij} \leq a_{ij} \leq \overline{a}_{ij}) \, \& \, \left(\underline{c}_i \leq \sum_{j:(i,j) \in I} a_{ij} \cdot x_j \leq \overline{c}_i \right) \right). \tag{10}$$

By separating the term containing a_{ij} in the second double inequality, we get an equivalent expression

$$\underline{c}_i \leq a_{ij} \cdot x_j + \sum_{k \neq j} a_{ik} \cdot x_k \leq \overline{c}_i,$$

or, equivalently,

$$\underline{c}_i - \sum_{k \neq j} a_{ik} \cdot x_k \leq a_{ij} \cdot x_j \leq \overline{c}_i - \sum_{k \neq j} a_{ik} \cdot x_k. \tag{11}$$

We know that $x_j \neq 0$, so we have either $x_j > 0$ or $x_j < 0$. Let is consider these two cases one by one.

2.3.1°. When $x_j > 0$, the inequality (11) takes an equivalent form

$$\frac{\underline{c}_i - \sum\limits_{k \neq j} a_{ik} \cdot x_k}{x_j} \leq a_{ij} \leq \frac{\overline{c}_i - \sum\limits_{k \neq j} a_{ik} \cdot x_k}{x_j}, \tag{12}$$

i.e., that a_{ij} belongs to the corresponding interval. So, the formula (10) means that there exists a value a_{ij} which belongs both to the interval $[\underline{a}_{ij}, \overline{a}_{ij}]$ and to the interval described by the formula (12). In other words, the formula (10) means that these two intervals have a common point.

One can easily check that two intervals $[\underline{p}, \overline{p}]$ and $[\underline{q}, \overline{q}]$ have a common point z if and only if the lower endpoint of each interval does not exceed the upper endpoint of the other interval, i.e., if and only if $\underline{p} \leq \overline{q}$ and $\underline{q} \leq \overline{p}$. Indeed:

- if there is a value z for which $\underline{p} \leq z \leq \overline{p}$ and $\underline{q} \leq z \leq \overline{q}$, then $\underline{p} \leq z \leq \overline{q}$ implies $\underline{p} \leq \overline{q}$ and $\underline{q} \leq z \leq \overline{p}$ implies $\underline{q} \leq \overline{p}$;

- vice versa, if there two inequalities are satisfied, then we can take $z = \max(\underline{p}, \underline{q})$; clearly, $z \geq \underline{p}$ and $z \leq \underline{q}$, so to conclude that z belongs to both intervals, we need to check that $z \leq \overline{p}$ and $z \leq \overline{q}$; indeed, we either have $z = \underline{p}$ or $z = \underline{q}$; in both cases, we have $z \leq \overline{p}$ and $z \leq \overline{q}$.

Based on this condition, the existence of the desired value a_{ij} is equivalent to the satisfaction of the following two inequalities:

$$\underline{a}_{ij} \le \frac{\overline{c}_i - \sum\limits_{k \ne j} a_{ik} \cdot x_k}{x_j}; \tag{13a}$$

$$\frac{\underline{c}_i - \sum\limits_{k \ne j} a_{ik} \cdot x_k}{x_j} \le \overline{a}_{ij}. \tag{13b}$$

Multiplying both inequalities by a positive number x_j, we get equivalent inequalities

$$\underline{a}_{ij} \cdot x_j \le \overline{c}_i - \sum\limits_{k \ne j} a_{ik} \cdot x_k; \tag{14a}$$

$$\underline{c}_i - \sum\limits_{k \ne j} a_{ik} \cdot x_k \le \overline{a}_{ij} \cdot x_j. \tag{14b}$$

Moving terms from one side to another, we get an equivalent inequality

$$\underline{c}_i - \overline{a}_{ij} \cdot x_j \le \sum\limits_{k \ne i} a_{ik} \cdot x_k \le \overline{c}_i - \underline{a}_{ij} \cdot x_j, \tag{15}$$

i.e., the desired form (9) with new bounds $\underline{c}_i - \overline{a}_{ij} \cdot x_j$ and $\overline{c}_i - \underline{a}_{ij} \cdot x_j$ instead of the previous bounds $\underline{c}_i$ and $\overline{c}_i$.

2.3.2°. When $x_j < 0$, the inequality (11) takes an equivalent form

$$\frac{\overline{c}_i - \sum\limits_{k \ne j} a_{ik} \cdot x_k}{x_j} \le a_{ij} \le \frac{\underline{c}_i - \sum\limits_{k \ne j} a_{ik} \cdot x_k}{x_j}, \tag{16}$$

i.e., that a_{ij} belongs to the corresponding interval. So, the formula (10) means that there exists a value a_{ij} which belongs both to the interval $[\underline{a}_{ij}, \overline{a}_{ij}]$ and to the interval described by the formula (16).

Similarly to Part 2.3.1, the existence of the desired value a_{ij} is equivalent to the satisfaction of the following two inequalities:

$$\underline{a}_{ij} \le \frac{\underline{c}_i - \sum\limits_{k \ne j} a_{ik} \cdot x_k}{x_j}; \tag{17a}$$

$$\frac{\overline{c}_i - \sum\limits_{k \ne j} a_{ik} \cdot x_k}{x_j} \le \overline{a}_{ij}. \tag{17b}$$

Multiplying both inequalities by a negative number x_j, we get equivalent inequalities

$$\overline{a}_{ij} \cdot x_j \le \overline{c}_i - \sum\limits_{k \ne j} a_{ik} \cdot x_k; \tag{18a}$$

$$\underline{c}_i - \sum\limits_{k \ne j} a_{ik} \cdot x_k \le \underline{a}_{ij} \cdot x_j. \tag{18b}$$

Moving terms from one side to another, we get an equivalent inequality

$$\underline{c}_i - \underline{a}_{ij} \cdot x_j \le \sum\limits_{k \ne i} a_{ik} \cdot x_k \le \overline{c}_i - \overline{a}_{ij} \cdot x_j, \tag{19}$$

i.e., the desired form (9) with new bounds $\underline{c}_i - \underline{a}_{ij}$ and $\overline{c}_i - \overline{a}_{ij}$ instead of the previous bounds $\underline{c}_i$ and $\overline{c}_i$.

2.4°. Let us now consider the case of the universal quantifier, for which the original condition takes the form

$$\forall a_{ij} \left((\underline{a}_{ij} \leq a_{ij} \leq \overline{a}_{ij}) \Rightarrow \left(\underline{c}_i \leq \sum_{j:(i,j)\in I} a_{ij} \cdot x_j \leq \overline{c}_i \right) \right). \tag{20}$$

Similarly to Part 2.3 of this proof, the double inequality in the conclusion part of the formula (20) is equivalent to a_{ij} being in an appropriate interval $[\underline{p}, \overline{p}]$: interval (12) when $x_j > 0$ and interval (16) when $x_j < 0$.

In both cases, the implication (20) means that all the points from the interval $[\underline{a}_{ij}, \overline{a}_{ij}]$ belong to the interval $[\underline{p}, \overline{p}]$. One can easily check that this requirement is equivalent to $\underline{p} \leq \underline{a}_{ij}$ and $\overline{a}_{ij} \leq \overline{p}$.

For the case of $x_j > 0$ and interval (12), this means that

$$\frac{\underline{c}_i - \sum_{k \neq j} a_{ik} \cdot x_k}{x_j} \leq \underline{a}_{ij}; \tag{21a}$$

$$\overline{a}_{ij} \leq \frac{\overline{c}_i - \sum_{k \neq j} a_{ik} \cdot x_k}{x_j}. \tag{21b}$$

Multiplying both sides by x_j and moving terms around, we get an equivalent double inequality

$$\underline{c}_i - \underline{a}_{ij} \cdot x_j \leq \sum_{k \neq i} a_{ik} \cdot x_k \leq \overline{c}_i - \overline{a}_{ij} \cdot x_j. \tag{22}$$

Here, two situations are possible:

- it is possible that $\underline{c}_i - \underline{a}_{ij} \cdot x_j > \overline{c}_i - \overline{a}_{ij} \cdot x_j$; in this case, the double inequality (22) cannot be satisfied, so we can conclude that the given vector x is not a solution to the given interval-quaantifier linear system;

- otherwise, if $\underline{c}_i - \underline{a}_{ij} \cdot x_j \leq \overline{c}_i - \overline{a}_{ij} \cdot x_j$, then we have a new double inequality of the desired form (9).

In the case of $x_j < 0$, we similarly get an equivalent double inequality

$$\underline{c}_i - \overline{a}_{ij} \cdot x_j \leq \sum_{k \neq i} a_{ik} \cdot x_k \leq \overline{c}_i - \underline{a}_{ij} \cdot x_j. \tag{23}$$

Here:

- it is possible that $\underline{c}_i - \overline{a}_{ij} \cdot x_j > \overline{c}_i - \underline{a}_{ij} \cdot x_j$, then we can conclude that the given vector x is not a solution to the given interval-quantifier linear system;

- otherwise, we have a new double inequality of the desired form (9).

Reduction is proven.

2.5°. To complete the proof, let us show that the above algorithm takes time $O(m \cdot n)$. Indeed, this algorithm consists of $m \cdot (n+1) = O(m \cdot n)$ stages – as many stages as there are quantifiers, and each stage takes the same finite number of computational steps. Thus, the overall computation time is

$$\text{const} \cdot O(m \cdot n) = O(m \cdot n).$$

The proposition is proven.

Examples: description.   To illustrate our algorithm, let us give two simple examples: when x is a solution and when x is not a solution. In both cases, we will have $n = m = 1$, and σ_1 is equality, so the system consists of a single equality $a_{11} \cdot x_1 = b_1$. We are looking for the tolerance solution, i.e., we are checking whether the following formula holds:

$$\forall a_{11} \in \boldsymbol{a}_{11} \, \exists b_1 \in \boldsymbol{b}_1 \, (a_{11} \cdot x_1 = b_1).$$

In both examples, we take $\boldsymbol{a}_{11} = [1, 2] = \boldsymbol{b}_1 = [1, 2]$.

In the first example, we take $x_1 = 1$; in this case, the desired formula is clearly true: for each $a_{11} \in \boldsymbol{a}_{11}$, we can take $b_1 = a_{11}$. In the second example, we take $x_1 = 2$. In this case, for $a_{11} = 2 \in \boldsymbol{a}_{11}$, we have $a_{11} \cdot x_1 = 4 \notin \boldsymbol{b}_1$. Let us apply our algorithm to these two examples.

Applying our algorithm to Example 1.   In this example, there are no zero values x_j, so we skip this step.

According to the above algorithm, we start by defining $a_{10} \stackrel{\text{def}}{=} b_1$ and taking $x_0 = -1$. In these notations, the original problem has the form

$$\forall a_{11} \in \boldsymbol{a}_{11} \, \exists a_{10} \in \boldsymbol{a}_{10} \, (a_{10} \cdot x_0 + a_{11} \cdot x_1 = 0).$$

Then, we describe the corresponding condition $a_{10} \cdot x_0 + a_{11} \cdot x_1 = 0$ as a double inequality:

$$\underline{c}_1 = 0 \leq a_{10} \cdot x_0 + a_{11} \cdot x_1 \leq \overline{c}_1 = 0.$$

We now eliminate quantifiers, starting with the innermost quantifier $\exists a_{10}$. In the beginning, none of the quantifiers are eliminated, so $I = \{(1, 0), (1, 1)\}$. According to the above algorithm, since the corresponding variable is negative $x_0 = -1 < 0$, eliminating the existential quantifier corresponding to results in the following double inequality:

$$\underline{c}_1 - \underline{a}_{10} \cdot x_0 \leq a_{11} \cdot x_1 \leq \overline{c}_1 - \overline{a}_{10} \cdot x_0,$$

i.e., computing the new values of the bounds $\underline{c}_1 - \underline{a}_{10} \cdot x_0 = 0 - 1 \cdot (-1) = 1$ and $\underline{c}_1 - \overline{a}_{10} \cdot x_0 = 0 - 2 \cdot (-1) = 2$, the inequality

$$\underline{c}_1 = 1 \leq a_{11} \cdot x_1 \leq \overline{c}_1 = 2.$$

Now, the set of not-yet-eliminated quantifiers I consists of only one pairs of indices $I = \{(1, 1)\}$. The corresponding quantifier is universal, and the value of the corresponding variable $x_1 = 1$ is positive, so eliminating this quantifier leads to the inequality

$$\underline{c}_1 - \underline{a}_{11} \cdot x_1 \leq 0 \leq \overline{c}_1 - \overline{a}_{11} \cdot x_1.$$

Here, $\underline{c}_1 - \underline{a}_{11} \cdot x_1 = 1 - 1 \cdot 1 = 0$ and $\overline{c}_1 - \overline{a}_{11} \cdot x_1 = 2 - 2 \cdot 1 = 0$, so the inequality is satisfied. Thus, the given value $x_1 = 1$ is a solution to the given interval-quantifier linear system.

Applying our algorithm to Example 2.   The only difference between Example 1 and Example 2 is that in Example 1, we had $x_1 = 1$, while in Example 2, we have $x_1 = 2$. Hence, for Example 2, the above algorithm follows the same steps until, in the process of eliminating the universal quantifier, it arrives at the inequality

$$\underline{c}_1 - \underline{a}_{11} \cdot x_1 \leq 0 \leq \overline{c}_1 - \overline{a}_{11} \cdot x_1.$$

Here, $\underline{c}_1 - \underline{a}_{11} \cdot x_1 = 1 - 1 \cdot 2 = -1$ and $\overline{c}_1 - \overline{a}_{11} \cdot x_1 = 2 - 2 \cdot 2 = -2$. So, the lower bound (-1) is larger than the upper bound (-2). Thus, we conclude that the given value $x_1 = 2$ is *not* a solution to the given interval-quantifier linear system.

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