Minimax Portfolio Optimization under Interval Uncertainty

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Abstract

In the 1950s, Markowitz proposed to combine different investment instruments to design a portfolio that either maximizes the expected return under constraints on volatility (risk) or minimizes the risk under given expected return. Markowitz's formulas are still widely used in financial practice. However, these formulas assume that we know the exact values of expected return and variance for each instrument, and that we know the exact covariance of every two instruments. In practice, we only know these values with some uncertainty. Often, we only know the bounds of these values – i.e., in other words, we only know the intervals that contain these values. In this paper, we show how to select an optimal portfolio under such interval uncertainty.

1 Formulation of the Problem

Variety of investments. There are different ways to invest money: we can deposit the money in a bank, we can buy stocks or bonds, we can buy securities, derivatives, and other financial instruments. Most investments come with risk: stocks or bounds can decrease their values, companies can go bankrupt, etc.

Usually, the less risky investments – such as depositing money in a bank – are the least profitable, while the most profitable schemes – such as investing in promising start-ups – are the most risky ones.

Every investor has a certain tolerance to risk, so he/she would like select his/her investments so as to maximize the return within a given risk level. Sometimes, an investor needs to maintain a certain growth level for his/her investments; in this case, out of all possible investment strategies that guarantee such return rate, the investor would like to select an investment that minimizes the risk.

Investment portfolios as a way to minimize (leverage) risk. Historically, among the most profitable investments are investments in stocks of promising technological companies: investors who bought Microsoft or Apple stocks when these stocks became available increased their original investment many times. However, such potentially profitable investments carry high risk, since many promising companies fail.

How to maintain high return while minimizing the risk? A natural idea is that, instead of investing all the money into a single stock ("putting all the eggs into one basket"), we spread our investment between different independent stocks. While each of these stocks can still fail, it is highly improbable that all these stocks will fail. As a result, in such a strategy, the risk of losing all the money is much smaller.

How to describe an investment portfolio: reminder. On the qualitative level, portfolios are clearly better than investing all the money into a single financial instrument. It is desirable to select a portfolio that makes the maximal use of this leveraging idea. For that, we need to be able to describe such portfolios in precise terms.

To specify a portfolio, we need to decide which portion of our money to invest in different available instruments. Let us denote the overall number of available financial instruments by n, and let us denote the portion that we invest in the i-th instrument by $w_i \geq 0$. Different investments should cover the whole amount, i.e., we should have $\sum\limits_{i=1}^{n} w_i = 1$.

Let us denote by r_i the return of the i-th investment. When the w_i-th part of the original money is invested in the i-th instrument, then the return from this part is equal to $w_i \cdot r_i$ and thus, the overall return r per unit investment is equal to $r = \sum\limits_{i=1}^{n} w_i \cdot r_i$.

How to estimate the investment risk. The risk associated with investments is due to the fact that it is not possible to predict the return r_i of each investment i. We can observe how this investment fared in the past; usually, in some years, this instrument grew, in others, grew less or even decreased in value. We can count the numbers of years with different increase and thus, estimate the probabilities of different return values. In other words, we can view the return r_i of the i-th instrument as a random variable – a variable that may take different values with different probabilities.

The portfolio's return r is a linear combination of a large number of random variables r_i – i.e., a sum of a large number of random variables $w_i \cdot r_i$. The leverage works when the investments are reasonably independent, and when each of these investments gets a reasonably small portion of the overall amount. Thus, r is a sum of a large number of small independent random variables $w_i \cdot r_i$.

It is known that for large n, the distribution of the sum of n small independent random variables is close to Gaussian (normal). This fact is known as the *Central Limit Theorem*; this is the main reason why normal distributions are ubiquitous in nature; see, e.g., [4]. Thus, the portfolio's return r can be viewed as a normally distributed random variable.

To describe a normally distributed random variable r, it is sufficient to describe two parameters: its expected value $\mu = E[r]$ and its variance $\sigma^2 = E\left[(r-\mu)^2\right]$. Thus, these two parameters are sufficient to describe the behavior of an investment portfolio: the expected return μ and the standard deviation σ. In economics, the portfolio's standard deviation is also called its *volatility*.

To formulate and solve the corresponding optimization problem, we need to be able to describe these two parameters μ and σ^2 in terms of the allocations w_i and of the parameters describing individual investments and the relation between them. For the mean, the situation is simple: the mean of the linear combination is equal to the linear combination of the means. Thus, we have $\mu = \sum\limits_{i=1}^{n} w_i \cdot \mu_i$, where $\mu_i = E[r_i]$ is the expected return of the i-th investment. Therefore,

$$r - \mu = \sum_{i=1}^{n} w_i \cdot r_i - \sum_{i=1}^{n} w_i \cdot \mu_i = \sum_{i=1}^{n} w_i \cdot (r_i - \mu_i)$$

and hence,

$$(r-\mu)^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i \cdot w_j \cdot (r_i - \mu_i) \cdot (r_j - \mu_j).$$

So,

$$\sigma^2 = E\left[(r-\mu)^2\right] = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i \cdot w_j \cdot \sigma_{ij},$$

where

$$\sigma_{ij} \stackrel{\text{def}}{=} E\left[(r_i - \mu_i) \cdot (r_j - \mu_j)\right]$$

is the corresponding covariance matrix.

Summarizing: to predict the expected return and volatility of each portfolio, we need to know the expected returns μ_i of each instrument and the covariance matrix σ_{ij} describing the volatility of individual instruments and relation between these instruments.

Markowitz's result: main assumptions and formulation of the problems. The main assumption behind the original Markowitz paper is that we do know the exact values of the quantities μ_i and σ_{ij}. Under this assumption, we can formulate the following two reasonable problems.

The first problem is related to the fact that each investor has a certain tolerance to risk. In precise terms, for each investor, there is the maximum value of volatility σ_0 that this investor can tolerate. Within this limit, we need to select a portfolio with the largest possible value of expected return μ. Of course, the larger risk an investor tolerates, potentially the larger the expected return, so it makes sense to always select a portfolio with the largest possible value of volatility. In precise terms, we thus need to solve the following problem:

$$\text{Maximize} \quad \sum_{i=1}^{n} w_i \cdot \mu_i$$

under the constraints

$$\sum_{i=1}^{n} \sum_{j=1}^{n} w_i \cdot w_j \cdot \sigma_{ij} = \sigma_0^2;$$

$$\sum_{i=1}^{n} w_i = 1.$$

In some situation, the investor is interested in achieving a certain level of average return μ_0. In this case, among all the portfolios that guarantee this level of expected return, we need to select a portfolio that minimizes the risk. In precise terms, we thus need to solve the following problem:

$$\text{Minimize} \quad \sum_{i=1}^{n} \sum_{j=1}^{n} w_i \cdot w_j \cdot \sigma_{ij}$$

under the constraints

$$\sum_{i=1}^{n} w_i \cdot \mu_i = \mu_0;$$

$$\sum_{i=1}^{n} w_i = 1.$$

Markowitz's result: algorithms. To solve each of these constraint optimization problems, we can the Lagrange multiplier method to reduce each of these problem to an easy-to-solve unconstrained optimization problem. For the first optimization problem, the Lagrange multiplier method leads to the problem of optimizing the function

$$\sum_{i=1}^{n} w_i \cdot \mu_i - \alpha \cdot \left(\sum_{i=1}^{n} \sum_{j=1}^{n} w_i \cdot w_j \cdot \sigma_{ij} - \sigma_0^2 \right) - \beta \cdot \left(\sum_{i=1}^{n} w_i - 1 \right),$$

(where α and β are Lagrange multipliers), or, equivalently, the problem of optimizing the function

$$\sum_{i=1}^{n} w_i \cdot \mu_i - \alpha \cdot \sum_{i=1}^{n} \sum_{j=1}^{n} w_i \cdot w_j \cdot \sigma_{ij} - \beta \cdot \sum_{i=1}^{n} w_i. \tag{1}$$

For the second problem, we get the problem of optimizing the function

$$\sum_{i=1}^n \sum_{j=1}^n w_i \cdot w_j \cdot \sigma_{ij} - \alpha' \cdot \left(\sum_{i=1}^n w_i \cdot \mu_i - \mu_0\right) - \beta' \cdot \left(\sum_{i=1}^n w_i - 1\right),$$

or, equivalently, the problem of optimizing

$$\sum_{i=1}^n \sum_{j=1}^n w_i \cdot w_j \cdot \sigma_{ij} - \alpha' \sum_{i=1}^n w_i \cdot \mu_i - \beta' \cdot \sum_{i=1}^n w_i.$$

If we divide this functional by α', we get the expression (1) with $\alpha = \dfrac{1}{\alpha'}$ and $\beta = -\dfrac{\beta}{\alpha'}$. Thus, for both problems, we need to optimize the expression (1).

Differentiating this expression by w_i and equating the derivatives to 0, we conclude that

$$2 \cdot \sum_{j=1}^n \sigma_{ij} \cdot w_j = \alpha \cdot \mu_i + \beta.$$

Thus, we have

$$w_i = \alpha \cdot w_i^{(1)} + \beta \cdot w_i^{(2)}, \tag{2}$$

where $w_i^{(k)}$ $(k = 1, 2)$ are the solutions to easy-to-solve systems of linear equations

$$2 \cdot \sum_{j=1}^n \sigma_{ij} \cdot w_j^{(1)} = \mu_i$$

and

$$2 \cdot \sum_{j=1}^n \sigma_{ij} \cdot w_j^{(2)} = 1.$$

The values α and β can be determined from the corresponding constraints. For the second problem, the constraints $\sum\limits_{i=1}^n w_i \cdot \mu_i = \mu_0$ and $\sum\limits_{i=1}^n w_i = 1$ take the form of an easy-to-solve system of two linear equations with two unknowns:

$$\alpha \cdot \mu^{(1)} + \beta \cdot \mu^{(2)} = \mu_0;$$

$$\alpha \cdot s^{(1)} + \beta \cdot s^{(2)} = 1,$$

where $\mu^{(k)} \stackrel{\text{def}}{=} \sum\limits_{i=1}^n w_i \cdot \mu_i^{(k)}$ and $s^{(k)} \stackrel{\text{def}}{=} \sum\limits_{i=1}^n w_i^{(k)}$. Once we find the values α and β, we can use the formula (2) to find the desired values w_i.

For the first problem, we get the constraints

$$\alpha^2 \cdot t_{11} + 2\alpha \cdot \beta \cdot t_{12} + \beta^2 \cdot t_{22} = \sigma_0^2; \tag{3}$$

$$\alpha \cdot s^{(1)} + \beta \cdot s^{(2)} = 1,$$

where $t_{k\ell} \stackrel{\text{def}}{=} \sum\limits_{i=1}^{n} \sum\limits_{j=1}^{n} w_i^{(k)} \cdot w_j^{(\ell)} \cdot \sigma_{ij}$. We can use the second equation to express β as a linear function of α, as

$$\beta = \frac{1}{s^{(2)}} - \alpha \cdot \frac{s^{(1)}}{s^{(2)}}. \tag{4}$$

Substituting this expression into the first equation of the system (3), we get an easy-to-solve quadratic equation, from which we can find α. Based on this α, we can use the formula (4) to find β and then the formula (2) to find the desired values w_i.

Remaining problem. Markowitz's formulas assume that we know the exact values of expected return and variance for each financial instrument, and that we know the exact covariance of every two instruments. In practice, we only know these values with some uncertainty.

Often, we only know the bounds of each of these values – i.e., in other words, we only know the intervals that contain these values. This means that instead of the exact values of the expected returns μ_i, we only know the bounds $\underline{\mu}_i \le \mu_i \le \overline{\mu}_i$, i.e., we only know the intervals $\left[\underline{\mu}_i, \overline{\mu}_i\right]$ that contain the actual (unknown) values μ_i. Similarly, instead of the exact values of σ_{ij}, we only know the bounds $\underline{\sigma}_{ij} \le \sigma_{ij} \le \overline{\sigma}_{ij}$, i.e., we only know the intervals $\left[\underline{\sigma}_{ij}, \overline{\sigma}_{ij}\right]$ that contain the actual (unknown) values σ_{ij}.

How can we select an optimal portfolio under such interval uncertainty? This is a question that we answer in this paper.

Comment. In addition to the intervals, we may have additional information about the values μ_i and σ_{ij}: partial information about the probabilities of different possible values from these intervals, fuzzy information about the degree of possibility of different values, etc. Several papers (see, e.g., [1, 3]) generalize Markowitz's ideas to situations when we have such additional information. In this paper, we assume that intervals is all we know.

2 Formulation of the Problems in Precise Terms

Towards the formulation of the first problem: constraint on volatility. In the first problem, we assume that an investor has a volatility threshold σ_0. In other words, we assume that the investor only considers investments for which the variance does not exceed the value σ_0^2.

In our situation, this means that we select the portfolio for which, for all possible combination of values $\sigma_{ij} \in \left[\underline{\sigma}_{ij}, \overline{\sigma}_{ij}\right]$, we must have

$$\sum_{i=1}^{n} \sum_{j=1}^{n} w_i \cdot w_i \cdot \sigma_{ij} \le \sigma_0^2.$$

Towards the formulation of the first problem: what do we maximize? In the first problem, we maximize the expected return $\mu = \sum\limits_{i=1}^{n} w_i \cdot \mu_i$. In the idealized case, when we know the exact values of expected returns μ_i for different instruments, we can use the above formula to uniquely determine the return corresponding to a given portfolio $(w_1, w_2, \ldots, w_n)$. In contrast, in the case of interval uncertainty, we may have different values of the return, depending on the actual values $\mu_i \in \left[\underline{\mu}_i, \overline{\mu}_i\right]$.

We argue that the investor should base his/her selections on the *smallest possible* return value. Indeed, this is the only expected return value that we can guarantee – anything above that comes with an additional risk that it will not happen.

Resulting formulation of the first problem. Thus, we arrive at the following "maximin" problem:

$$\text{Maximize} \quad \min_{\mu_i \in \left[\underline{\mu}_i, \overline{\mu}_i\right]} \sum_{i=1} w_i \cdot \mu_i$$

under the constraints

$$\sum_{i=1}^{n} \sum_{j=1}^{n} w_i \cdot w_j \cdot \sigma_{ij} \le \sigma_0^2 \text{ for all } \sigma_{ij} \in \left[\underline{\sigma}_{ij}, \overline{\sigma}_{ij}\right];$$

$$\sum_{i=1}^{n} w_i = 1.$$

Towards the formulation of the second problem. In the second problem, we want to guarantee the return rate $\mu = \sum\limits_{i=1}^{n} w_i \cdot \mu_i$ to be at least μ_0. The actual return rate depends on the values $\mu_i \in \left[\underline{\mu}_i, \overline{\mu}_i\right]$, so the only way to guarantee that the actual return rate is greater than or equal to μ_0 is to make sure that all possible values of $\mu = \sum\limits_{i=1}^{n} w_i \cdot \mu_i$ are greater than or equal to μ_0.

Under this constraint, we want to minimize the risk σ^2. Again, the actual value of the risk depends on the values $\sigma_{ij} \in \left[\underline{\sigma}_{ij}, \overline{\sigma}_{ij}\right]$. Minimizing the risk usually minimizing the *worst-case* risk, i.e., minimize the maximum value of the risk.

Resulting formulation of the second problem. Thus, we arrive at the following "minimax" problem:

$$\text{Minimize} \quad \max_{\sigma_{ij} \in \left[\underline{\sigma}_{ij}, \overline{\sigma}_{ij}\right]} \sum_{i=1}^{n} \sum_{j=1}^{n} w_i \cdot w_j \cdot \sigma_{ij}$$

under the constraints

$$\sum_{i=1} w_i \cdot \mu_i \text{ for all } \mu_i \in \left[\underline{\mu}_i, \overline{\mu}_i\right];$$

$$\sum_{i=1}^{n} w_i = 1.$$

3 Analysis of the Resulting Problems

Analyzing the first problem. In the first problem, the objective function is $\min\limits_{\mu_i \in \left[\underline{\mu}_i, \overline{\mu}_i\right]} \sum\limits_{i=1} w_i \cdot \mu_i$. Since all the allocations w_i are non-negative, the expression $\sum\limits_{i=1} w_i \cdot \mu_i$ is a monotonic (non-strictly increasing) function of the values μ_i. Thus, its minimum is attained when each value μ_i attains its smallest possible value $\underline{\mu}_i$. So, the objective function takes the form $\sum\limits_{i=1} w_i \cdot \underline{\mu}_i$.

The first constraint is that $\sum\limits_{i=1}^{n} \sum\limits_{j=1}^{n} w_i \cdot w_j \cdot \sigma_{ij} \le \sigma_0^2$ for all $\sigma_{ij} \in \left[\underline{\sigma}_{ij}, \overline{\sigma}_{ij}\right]$. This inequality is equivalent to

$$\max_{\sigma_{ij} \in \left[\underline{\sigma}_{ij}, \overline{\sigma}_{ij}\right]} \sum_{i=1}^{n} \sum_{j=1}^{n} w_i \cdot w_j \cdot \sigma_{ij} \le \sigma_0^2.$$

Since the allocations w_i are non-negative, the function $\sum\limits_{i=1}^{n} \sum\limits_{j=1}^{n} w_i \cdot w_j \cdot \sigma_{ij}$ is a monotonic (non-strictly increasing) function of the values σ_{ij}. Thus, its maximum is attained when each value σ_{ij} attains its largest possible value $\overline{\sigma}_{ij}$. So, the constraint takes the form $\sum\limits_{i=1}^{n} \sum\limits_{j=1}^{n} w_i \cdot w_j \cdot \overline{\sigma}_{ij} \le \sigma_0^2$.

Hence, the first problem takes the following form:

$$\text{Maximize } \sum_{i=1} w_i \cdot \underline{\mu}_i$$

under the constraints

$$\sum_{i=1}^{n} \sum_{j=1}^{n} w_i \cdot w_j \cdot \overline{\sigma}_{ij} \le \sigma_0^2;$$

$$\sum_{i=1}^{n} w_i = 1.$$

We have already mentioned that the larger the allowed risk, potentially the larger the resulting gain. Thus, when we are maximizing the gain, it makes sense to only consider situations in which the worst-case risk $\sum\limits_{i=1}^{n} \sum\limits_{j=1}^{n} w_i \cdot w_j \cdot \overline{\sigma}_{ij}$

takes the largest possible value σ_0^2. In other words, it is reasonable to consider the following optimization problem:

$$\text{Maximize} \quad \sum_{i=1} w_i \cdot \underline{\mu}_i$$

under the constraints

$$\sum_{i=1}^{n} \sum_{j=1}^{n} w_i \cdot w_j \cdot \overline{\sigma}_{ij} = \sigma_0^2;$$

$$\sum_{i=1}^{n} w_i = 1.$$

This is the first original Markowitz problem for $\mu_i = \underline{\mu}_i$ and $\sigma_{ij} = \overline{\sigma}_{ij}$. Thus, we can use the known algorithms for solving the first Markowitz problem for exactly known μ_i and σ_{ij} to solve a similar problem corresponding to interval uncertainty.

Analyzing the second problem. In the second problem, the objective function is $\max\limits_{\sigma_{ij} \in \left[\underline{\sigma}_{ij}, \overline{\sigma}_{ij}\right]} \sum\limits_{i=1}^{n} \sum\limits_{j=1}^{n} w_i \cdot w_j \cdot \sigma_{ij}$. Since all the allocations w_i are non-negative, the expression $\sum\limits_{i=1}^{n} \sum\limits_{j=1}^{n} w_i \cdot w_j \cdot \sigma_{ij}$ is a monotonic (non-strictly increasing) function of the values σ_{ij}. Thus, its maximum is attained when each value σ_{ij} attains its largest possible value $\overline{\sigma}_{ij}$. So, the objective function takes the form $\sum\limits_{i=1}^{n} \sum\limits_{j=1}^{n} w_i \cdot w_j \cdot \overline{\sigma}_{ij}$.

The first constraint is that $\sum\limits_{i=1}^{n} w_i \cdot \mu_i \geq \mu_0$ for all $\mu_i \in \left[\underline{\mu}_i, \overline{\mu}_i\right]$. This inequality is equivalent to

$$\min_{\mu_i \in \left[\underline{\mu}_i, \overline{\mu}_i\right]} \sum_{i=1}^{n} w_i \cdot \mu_i \geq \mu_0.$$

Since the allocations w_i are non-negative, the function $\sum\limits_{i=1}^{n} w_i \cdot \mu_i$ is a monotonic (non-strictly increasing) function of the values μ_i. Thus, its minimum is attained when each value μ_i attains its smallest possible value $\underline{\mu}_i$. So, the constraint takes the form $\sum\limits_{i=1} w_i \cdot \underline{\mu}_i \geq \mu_0$.

Hence, the second problem takes the following form:

$$\text{Minimize} \quad \sum_{i=1}^{n} \sum_{j=1}^{n} w_i \cdot w_j \cdot \overline{\sigma}_{ij}$$

under the constraints

$$\sum_{i=1} w_i \cdot \underline{\mu}_i \geq \mu_0;$$

$$\sum_{i=1}^{n} w_i = 1.$$

As we have mentioned earlier, the higher return we want, potentially the higher the corresponding risk. Thus, when we are minimizing the gain, it makes sense to only consider situations in which the worst-case gain $\sum_{i=1} w_i \cdot \underline{\mu}_i$ takes the smallest possible value μ_0. In other words, it is reasonable to consider the following optimization problem:

$$\text{Minimize} \quad \sum_{i=1}^{n} \sum_{j=1}^{n} w_i \cdot w_j \cdot \overline{\sigma}_{ij}$$

under the constraints

$$\sum_{i=1} w_i \cdot \underline{\mu}_i = \mu_0;$$

$$\sum_{i=1}^{n} w_i = 1.$$

This is the second original Markowitz problem for $\mu_i = \underline{\mu}_i$ and $\sigma_{ij} = \overline{\sigma}_{ij}$. Thus, we can use the known algorithms for solving the second Markowitz problem for exactly known μ_i and σ_{ij} to solve a similar problem corresponding to interval uncertainty.

4 How to Select an Optimal Portfolio in the Case of Interval Uncertainty: Resulting Recommendations

What information we have: reminder. We have n possible financial instruments. For each of the instruments i, we know the bounds $\underline{\mu}_i$ and $\overline{\mu}_i$ on the actual (unknown) return μ_i of this instrument: $\underline{\mu}_i \le \mu_i \le \overline{\mu}_i$.

Also, for each pairs of instruments i and j, we know the bounds $\underline{\sigma}_{ij}$ and $\overline{\sigma}_{ij}$ on the covariance σ_{ij} between these two instruments: $\underline{\sigma}_{ij} \le \sigma_{ij} \le \overline{\sigma}_{ij}$.

Comment. The original problem analyzed by Markowitz corresponds to the case when we know the exact values of all these quantities, i.e., when $\underline{\mu}_i = \overline{\mu}_i$ and $\underline{\sigma}_{ij} = \overline{\sigma}_{ij}$ for all i and j.

Two possible situations: reminder. Similarly to the original Markowitz problem, we consider two possible situations:

- In the first situation, we know the highest possible value of risk σ_0 tolerated by the investor. Under this constraint, we want to find a portfolio with the largest guaranteed rate of return.

- In the second situation, we want to guarantee the rate of return μ_0. Under this constraint, we want to find the portfolio with the smallest risk.

Our recommendation. The above analysis shows that in both situations, to find the optimal portfolio, we must solve the original Markowitz problem for $\mu_i = \underline{\mu}_i$ and $\sigma_{ij} = \overline{\sigma}_{ij}$.

Discussion. This mathematical recommendation makes perfect sense: we do not want to add additional risk, so we operate under the worst-case conditions. From the viewpoint of gain, the worst-case situation is when the gain is the smallest, i.e., when $\mu_i = \underline{\mu}_i$. From the viewpoint of risk, the worst-case situation is when the risk is the largest, i.e., when $\sigma_{ij} = \overline{\sigma}_{ij}$.

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