Newton's Laws: What is Their Operational Meaning?

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Abstract

Newton's mechanics is one of the most successful theories in the history of science; its success is based on three Newton's laws. At first glance, the Newton's laws that describe the relation between masses, forces, and accelerations are very clear and straightforward. However, the situation becomes more ambiguous if we take into account that the notions of mass and force are not operationally defined. In this paper, we describe the operational meaning of Newton's laws.

1 It Is Important to Reformulate Newton's Laws in Operational Terms: Formulation of the Problem

Original formulation of Newton's laws: reminder [1, 5]**.** The first Newton's law – law of inertia – states that if no force is acting on a body, this body retains its speed and direction of motion. The second law states that the force $\vec{F}$ is equal to the product of mass m and acceleration $\vec{a}$: $\vec{F} = m \cdot \vec{a}$. The third law states that if a body A acts on a body B with a force $\vec{F}$, then the body B acts on the body A with the force $-\vec{F}$.

Pedagogical problem: we need an operational reformulation. Of course, we have an intuitive notion of what is a mass and what is a force. However, for most people, these intuitive notions are somewhat vague, and to understand Newton's laws, we need to be able to provide a precise numerical meaning of these terms.

Without such operational meaning, Newton's laws sounds very abstract: there exist some precise notions of mass and force for which the above three laws hold. This is probably how some students understand these laws. If all we have is such an abstract formulation, it is no wonder that some students have trouble applying these laws to real-life problems.

Foundational problem: we need an operational reformulation. Operational reformulation is needed also because Newton's laws aim at describing the physical world. How do we know that these laws are valid? How can we check that these laws are not valid? For example, what do physicists mean when they claim that Newton's laws are not valid in relativistic mechanics?

When Newton's laws are formulated in the above abstract form, without providing any operational meaning for mass and force, then it is not clear how to check whether the given experimental data supports these laws or not. To be able to do that, we need to reformulate Newton's laws in operational terms, i.e., in terms of observations.

2 Reformulating Newton's Laws in Operational Terms: A Straightforward Approach

First Newton's law: a straightforward reformulation. The first Newton's law was actually first formulated by Galileo [3]. This law has a straightforward operational interpretation: if we have only one body A, then its acceleration is zero: $\vec{a}_A = 0$.

Of course, in reality, we always have some other bodies in the Universe, but if these bodies are sufficiently far away, we can safely assume that their influence is negligible. We can therefore reformulate this law in the following form: when we move a body A further and further away from all other bodies, its acceleration gets closer and closer to 0.

Comment. This reformulation assumes that the force between the bodies decreases as the distance between them increases. This is definitely true for usual forces such as gravity or electromagnetic forces, but it is worth mentioning that not all forces are like that: for example, the force acting between the two quarks *increases* when the distance between them increases; see, e.g., [1].

Second Newton's law: does it mean anything? By itself, the second Newton's law can be simply viewed as a *definition* of the force: once we know how to define masses, we can then define the force $\vec{F}$ as the product $m \cdot \vec{a}$. Thus, no matter how bodies move, the second law is always satisfied, if we simply take $\vec{F} \stackrel{\text{def}}{=} m \cdot \vec{a}$.

From this viewpoint, the second law does not tell us anything at all. Okey, there is an implicit assumption of *determinism* here, that if we place the same bodies at same locations with same initial velocities, then we will observe the same accelerations, but from the second law itself, we cannot conclude anything beyond that.

Comments.

- It is also usually implicitly assumed that a finite number of parameters is sufficient to describe a body, its position, velocity, and orientation, and that once we know the values of all these parameters, we can uniquely determine all the forces.

- Determinism is what distinguishes Newton's mechanics from quantum physics, where we can only predict *probabilities* of different measurement results, but not the measurement results themselves.

What if we also take into account the third law? If we also take the third law into account, then the situation changes. Literally, the third law says that for every two bodies A and B, the force $\vec{F}_{A|B}$ with which the body B acts on the body A and the force $\vec{F}_{B|A}$ describing the influence of the body A on the body B are related by the formula $\vec{F}_{B|A} = -\vec{F}_{A|B}$. If we substitute the definition $\vec{F} = m \cdot \vec{a}$ into this formula, we conclude that in the situation when we only have two bodies A and B, the following is true: $m_B \cdot \vec{a}_{B|A} = -m_A \cdot \vec{a}_{A|B}$, where m_A and m_B are the masses of the bodies A and B, and $\vec{a}_{A|B}$ and $\vec{a}_{B|A}$ are their accelerations.

We still do not have an operational definition of mass, so the above rule can be reformulated as follows: it is possible to assign, to every body A, a number m_A so that in every situation in which there are only two bodies A and B, we have $m_B \cdot \vec{a}_{B|A} = -m_A \cdot \vec{a}_{A|B}$. How can check this possibility experimentally?

How to check the third law? One thing we can check right away: that the vectors $\vec{a}_{A|B}$ and $\vec{a}_{B|A}$ have the same direction.

Since these two vectors have the same direction, we can define their ratio $r_{A|B} \stackrel{\text{def}}{=} \frac{\vec{a}_{B|A}}{\vec{a}_{B|A}}$ as a real number for which $\vec{a}_{B|A} = r_{A|B} \cdot \vec{a}_{A|B}$. According to the above formula, this (observable) ratio has the form $r_{A|B} = -\frac{m_A}{m_B}$.

So, the question of how to reformulate the third law in operational terms can be described as follows:

- for every two bodies A and B, we can experimentally determine the ratios $r_{A|B}$;
- we want to check whether there exist values m_A for which $r_{A|B} = -\frac{m_A}{m_B}$ for all pairs (A, B).

One can easily see that if such values m_A exist, then for every three bodies A, B, and C, we have $r_{A|C} = -r_{A|B} \cdot r_{B|C}$. Vice versa, if this property is satisfied, then we can find appropriate m_A: for example, we can fix some object A_0 and then take $m_A = r_{A|A_0}$. Indeed, in this case, for $C = A_0$, we have $r_{A|A_0} = -r_{A|B} \cdot r_{B|A_0}$, i.e., $m_A = -r_{A|B} \cdot m_B$ and thus, $r_{A|B} = -\frac{m_A}{m_B}$.

Comment. An additional implicit assumption behind Newton's physics is that in general, the body mass does not change with time. To be more precise, it may change – e.g., for a rocket flying to the Moon – but this is because the original rocket consisted of two parts: the rocket itself and the fuel. Each part retains its mass, but the parts become separated as the fuel flies away.

This constancy of mass is what separates Newton's mechanics from special relativity, where a body's mass changes with the body's speed v as $m = \dfrac{m_0}{\sqrt{1 - \dfrac{v^2}{c^2}}}$, where c is the speed of light.

Straightforward interpretation of Newton's laws. Thus, the straightforward interpretation of Newton's laws is as follows.

- If the body A is the only body in the world, then its acceleration is equal to 0: $\vec{a}_A = 0$.
- For every body A, its acceleration $\vec{a}_A$ is uniquely determined by the positions, velocities, and orientations of this body A and of all other bodies.
- Let $\vec{a}_{A|B}$ denote the acceleration of the body A in the situation when the only other body present is body B. In this case:
  - for every two bodies A and B, the vectors $\vec{a}_{A|B}$ and $\vec{a}_{B|A}$ are collinear, i.e., $\vec{a}_{B|A} = r_{A|B} \cdot \vec{a}_{A|b}$ for some scalar $r_{A|B}$;
  - for every three bodies A, B, and C, we have $r_{A|C} = -r_{A|B} \cdot r_{B|C}$.

3 Reformulating Newton's Laws in Operational Terms: An Additional Property – Additivity of Forces

The above reformulation is rather weak. One can see that in this reformulation, the first and the third laws are meaningful, while the second law – which is usually portrayed as the main law of Newton's physics – practically disappears: it is reduced simply to determinism.

So, how did Newton make predictions? If this is the case, if the second law does not have any serious meaning, then how come Newton succeeded in getting so many observable predictions out of his laws? Yes, he used a specific formula for the gravity force, but this is not sufficient: this would be sufficient for situations when we have only two bodies, but Newton also analyzed situations with three or more bodies. How did he do it?

Enter additivity of forces. In his analysis, Newton also also used another property, a property which he did not explicitly formulate as one of his laws, but which is very important for making predictions: the implicit property of *additivity of forces*. Namely, he assumes that in the presence of several bodies, a force acting on a given body A is equal to the sum of the forces coming from all these bodies.

In precise terms, the force $\vec{F}_{A|B,\ldots,C}$ that bodies $B,\ldots,C$ exert on body A is equal to the sum of the forces $\vec{F}_{A|B}$, ..., $\vec{F}_{A|C}$ that the body A would experience in the presence of only one other body B, ..., or C:

$$\vec{F}_{A|B,\ldots,C} = \vec{F}_{A|B} + \ldots + \vec{F}_{A|C}.$$

Comment. This additivity property is sometimes explicitly mentioned as an important part of the second Newton's Law – for example, it is listed as such on the Wikipedia page on Newton's laws – but Newton never explicitly formulated this property.

Let us reformulate additivity in operational terms. According to the second Newton's law, $\vec{F}_{A|B,\ldots,C} = m_A \cdot \vec{a}_{A|B,\ldots,C}$, $\vec{F}_{A|B} = m_A \cdot \vec{a}_{A|B}$, ..., $\vec{F}_{A|C} = m_A \cdot \vec{a}_{A|C}$, where $\vec{a}$ are corresponding accelerations. Substituting these expressions into the above formula and dividing both sides by the common factor m_A, we get the desired reformulation.

Operational reformulation of additivity of forces. The acceleration $\vec{a}_{A|B,\ldots,C}$ that bodies $B,\ldots,C$ exert on body A is equal to the sum of the accelerations $\vec{a}_{A|B}$, ..., $\vec{a}_{A|C}$ that the body A would experience in the presence of only one other body B, ..., or C:

$$\vec{a}_{A|B,\ldots,C} = \vec{a}_{A|B} + \ldots + \vec{a}_{A|C}.$$

Comment. In the appendix, we describe how to tell when a function of many variables can be represented as a sum of such pairwise expressions.

4 What We Can Conclude Based on Additivity of Forces

First conclusion: momentum is preserved. For each body A, due to additivity, we have $m_A \cdot \vec{A} = \sum\limits_{B\neq A} \vec{F}_{A|B}$. If we add up all both sides corresponding to all the bodies A, we will be able to conclude that

$$\sum_A m_A \cdot \vec{a}_A = \sum_A \sum_B \vec{F}_{A|B}.$$

In the right-hand side of this formula, each pair of objects (A,B) occurs twice: as $\vec{F}_{A|B}$ and as $\vec{F}_{B|A}$. Due to the third Newton's law, $\vec{F}_{A|B} + \vec{F}_{B|A} = 0$. Thus, $\sum\limits_A \sum\limits_B \vec{F}_{A|B} = 0$ and, therefore, $\sum\limits_A m_A \cdot \vec{a}_A = 0$. Each acceleration $\vec{a}_A$ is a time derivative of the corresponding velocity $\vec{v}_A$. Thus,

$$\frac{d}{dt}\left(\sum_A m_A \cdot \vec{v}_A\right) = 0.$$

In other words, the *momentum* $\sum_A m_A \cdot \vec{v}_A$ does not change with time.

Comments.

- An alternative derivation of the momentum preservation property is given in [2].
- It is worth mentioning that the momentum is preserved in special relativity as well, the difference is that in special relativity theory, as we have mentioned earlier, the mass changes when velocity changes.

Second conclusion: additivity of mass. Let us assume that we have two bodies A and B travelling together, with the same acceleration $\vec{a}$. We can view this situation in two different ways:

- as two different bodies A and B each travelling with the acceleration $\vec{a}$, or
- as a single composite body AB travelling with an acceleration $\vec{a}$.

In the first case, for the body A, the second Newton's law has the form $m_A \cdot \vec{a} = \vec{F}_A$, where, due to additivity, the force $\vec{F}$ is the sum of two components: the force $\vec{F}_{A|B}$ coming from the body B and the force $\vec{F}_{A|X}$ coming from all other bodies X: $m_A \cdot \vec{a} = \vec{F}_{A|B} + \vec{F}_{A|X}$. Similarly, we have $m_B \cdot \vec{a} = \vec{F}_{B|A} + \vec{F}_{B|X}$. By adding these two formulas and by taking into account that, due to the third Newton's law, $\vec{F}_{A|B} + \vec{F}_{B|A} = 0$, we conclude that

$$(m_A + m_B) \cdot \vec{a} = \vec{F}_{A|X} + \vec{F}_{B|X}.$$

On the other hand, in the second interpretation, we have a single composite body AB of some mass m_{AB} which is accelerating due to forces $\vec{F}_{A|X}$ and $\vec{F}_{B|X}$ acting on this composite body. Due to additivity of forces, the overall, force acting on the composite body AB is equal to $\vec{F}_{A|X} + \vec{F}_{B|X}$. Thus, for this composite body, the second Newton's Law takes the form

$$m_{AB} \cdot \vec{a} = \vec{F}_{A|X} + \vec{F}_{B|X}.$$

By comparing the formulas corresponding to the two possible interpretation of this situation, we conclude that

$$m_{AB} = m_A + m_B.$$

In other words, mass is *additive* in the sense that the mass of the composite body is equal to the sum of the masses of its components.

Comment. This argument is similar to the one given in [2].

Deriving laws of gravity (almost). Since one of the main original successes of Newton's physics was the description of the motion caused by the gravitational forces, it is worth mentioned that the formula for the gravitational force can be – almost uniquely – determined based on additivity.

Indeed, the value of the gravitational force $\vec{F}_{A|B}$, by definition, is determined only by the masses of the bodies m_A and m_B and by the mutual location $\vec{r}$ of these two bodies: $\vec{F}_{A|B} = \vec{F}(m_A, m_B, \vec{r})$ for some vector-valued function $\vec{F}$.

If the body B consists of two parts B_1 and B_2 of masses, correspondingly, m_1 and m_2, then we can view this situation in two different ways:

- we can treat the body B as two different bodies B_1 and B_2 each affecting the body A, or
- we can treat B as a single body affecting the body A.

In the first case, due to additivity of forces, the force acting on the body A is equal to the sum

$$\vec{F}_{A|B} = \vec{F}_{A|B_1} + \vec{F}_{A|B_2} = \vec{F}(m_A, m_1, \vec{r}) + \vec{F}(m_A, m_2, \vec{r}).$$

On the other hand, in the second interpretation, due to the additivity of masses $m_B = m_1 + m_2$, this same force has the form

$$\vec{F}_{A|B} = \vec{F}(m_A, m_B, \vec{r}) = \vec{F}(m_A, m_1 + m_2, \vec{r}).$$

By comparing the formulas corresponding to the two possible interpretation of this situation, we conclude that

$$\vec{F}(m_A, m_1 + m_2, \vec{r}) = \vec{F}(m_A, m_1, \vec{r}) + \vec{F}(m_A, m_2, \vec{r}).$$

In other words, for every m_A and $\vec{r}$ and for each spatial component i, the function $f(m) \stackrel{\text{def}}{=} F_i(m_A, m, \vec{r})$ satisfies the additivity property $f(m_1 + m_2) = f(m_1) + f(m_2)$.

One can easily see that the only continuous function with this property is a function $f(m) = k \cdot m$, where $k \stackrel{\text{def}}{=} f(1)$. Indeed, this is trivially true for $m = 1$. For $m = \dfrac{1}{n}$, we have

$$f(1) = f\left(\frac{1}{n}\right) + \ldots + f\left(\frac{1}{n}\right) \quad (n \text{ times}),$$

so $f(1) = n \cdot f\left(\dfrac{1}{n}\right)$ and $f\left(\dfrac{1}{n}\right) = \dfrac{1}{n} \cdot f(1) = k \cdot m$.

For rational $m = \dfrac{p}{n}$, we have

$$f(m) = f\left(\frac{p}{n}\right) = f\left(\frac{1}{n}\right) + \ldots + f\left(\frac{1}{n}\right) \quad (p \text{ times}),$$

so

$$f(m) = f\left(\frac{p}{n}\right) = p \cdot f\left(\frac{1}{n}\right) = p \cdot \frac{1}{n} \cdot k = \frac{p}{n} \cdot k = k \cdot m.$$

Since every real number can be represented as a limit of rational numbers, and $f(m) = k \cdot m$ for all rational numbers, continuity implies that $f(m) = k \cdot m$ for all values m. Thus,

$$\vec{F}(m_A, m_B, \vec{r}) = \vec{f}(m_A, \vec{r}) \cdot m_B,$$

where we denoted $\vec{f}(m_A, \vec{r}) \stackrel{\text{def}}{=} \vec{F}(m_A, 1, \vec{r})$.

Similarly, if the body A consists of two parts A_1 and A_2 with masses m_1 and m_2, then we can view this situation in two different ways:

- as two different bodies A_1 and A_2 both affected by B, or
- as a single composite body A affected by the body B.

In the first case, due to the additivity of forces, the overall force acting on the body A is equal to $\vec{F}(m_1, m_B, \vec{r}) + \vec{F}(m_2, m_B, \vec{r})$. In the second case, this force is equal to $\vec{F}(m_A, m_B, \vec{r}) = \vec{F}(m_1 + m_2, m_B, \vec{r})$. By comparing these two expressions for the same force, we conclude that

$$\vec{F}(m_1 + m_2, m_B, \vec{r}) = \vec{F}(m_1, m_B, \vec{r}) + \vec{F}(m_2, m_B, \vec{r}).$$

Substituting the expression $\vec{F}(m_A, m_B, \vec{r}) = \vec{f}(m_A, \vec{r}) \cdot m_B$ into this formula and dividing both sides of the resulting equality by m_B, we conclude that

$$\vec{f}(m_1 + m_2, \vec{r}) = \vec{f}(m_1, \vec{r}) + \vec{f}(m_2, \vec{r}).$$

Thus, similar arguments lead to $\vec{f}(m_A, \vec{r}) = m_A \cdot \vec{g}(\vec{r})$. Hence,

$$\vec{F}(m_A, m_B, \vec{r}) = m_A \cdot m_B \cdot \vec{g}(\vec{r})$$

for some function $\vec{g}(\vec{r})$.

Comment. It is worth mentioning that for this formula, the first Newton's law is automatically satisfied: when $m_B = 0$, we have $\vec{F} = 0$.

Deriving laws of gravity (cont-d). If we require that this expression be rotation-invariant, we can then conclude that $\vec{g}(\vec{r}) = \vec{r} \cdot h(r)$ for some function $h(r)$, where $r \stackrel{\text{def}}{=} |\vec{r}|$ is the distance between the two bodies.

Comment. For this formula, the third Newton's law is also automatically satisfied, since here, $\vec{F}_{B|A} = m_B \cdot m_A \cdot (-\vec{r}) \cdot h(r) = -\vec{F}_{A|B}$.

Deriving laws of gravity (final part). Finally, if we require that the dependence be scale-invariant, i.e., that a re-scaling of distances $r \to \lambda \cdot r$ (e.g., changing from meters to centimeters) will lead to the same formula for the force, but maybe after an appropriate re-scaling of force. In precise terms, this means that for every λ, there exists a value $a(\lambda)$ for which $h(\lambda \cdot r) = a(\lambda) \cdot h(r)$.

If we first re-scale by a factor of λ_1 (i.e., go from r to $r' = \lambda_1 \cdot r$), and then by a factor of λ_2 (i.e., go from r' to $r'' = \lambda_2 \cdot r' = \lambda \cdot r$, where $\lambda \stackrel{\text{def}}{=} \lambda_1 \cdot \lambda_2$), then we get

$$h(\lambda \cdot r) = h(\lambda_2 \cdot (\lambda_1 \cdot r)) = a(\lambda_2) \cdot h(\lambda_1 \cdot r) = a(\lambda_2) \cdot a(\lambda_1) \cdot h(r).$$

On the other hand, we have $h(\lambda \cdot r) = a(\lambda) \cdot h(r)$. By comparing these two formulas, we conclude that

$$a(\lambda) = a(\lambda_1 \cdot \lambda_2) = a(\lambda_1) \cdot a(\lambda_2).$$

This equation is similar to the one that we had before, except that now we have multiplications instead of additions. We can use $\ln(x)$ to reduce multiplication to addition. By taking logarithms of both sides, we get

$$\ell(\lambda_1 \cdot \lambda_2) = \ell(\lambda_1) + \ell(\lambda_2),$$

where we denoted $\ell(x) \stackrel{\text{def}}{=} \ln(a(x))$. For the function $A(X) \stackrel{\text{def}}{=} \ell(\exp(X)) = \ln(a(\exp(X)))$, we have $\ell(x) = A(\ln(x))$, so the above formula takes the form

$$A(\ln(\lambda_1 \cdot \lambda_2)) = A(\ln(\lambda_1)) + A(\ln(\lambda_2)).$$

Here, $\ln(\lambda_1 \cdot \lambda_2) = x_1 + x_2$, where $x_i \stackrel{\text{def}}{=} \ln(\lambda_i)$, so the formula takes the additivity form $A(x_1 + x_2) = A(x_1) + A(x_2)$.

We already know that in this case, $A(x) = \alpha \cdot x$ for some x. Thus, $\ell(x) = A(\ln(x)) = \alpha \cdot \ln(x)$, and $a(x) = \exp(\ell(x)) = \exp(\alpha \cdot \ln(x)) = x^\alpha$. Now, from $h(\lambda \cdot r) = a(\lambda) \cdot h(r)$, when $r = 1$, we get $h(x) = h(1) \cdot x^\alpha$, i.e., $h(r) = C \cdot r^\alpha$ for some values C and α. Therefore,

$$\vec{F}(m_A, m_B, \vec{r}) = C \cdot m_A \cdot m_B \cdot \vec{r} \cdot r^\alpha.$$

Comments.

- This is *almost* Newton's law describing gravity. To get exactly the Newton's law, we need to specify $\alpha = -3$.
- Similarly, if we define electrostatic forces as depending only on the additive charges q_A and q_B, then we get $\vec{F}_{A|B} = D \cdot q_A \cdot q_B \cdot \vec{r} \cdot r^\beta$ for some values D and β.

Acknowledgments. This work was supported in part by the National Science Foundation grants HRD-0734825 and HRD-1242122 (Cyber-ShARE Center of Excellence), and DUE-0926721.

The authors are thankful to all the participants of the IEEE Symposium on Computational Intelligence for Engineering Solutions CIES'2014 (Orlando, Florida, December 9–12, 2014) for valuable discussions.

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A Formalizing Additivity of Forces: How to Tell When a Function of Several Variables is Equal to the Sum of Pairwise Functions

Definition. *Assume that the integers from 1 to n are divided into several groups A, …, B. For a tuple $x_1, \ldots, x_n$ and for a group A, by x_A, we denote a sub-tuple consisting of all the values x_i with $i \in A$. We say that a function $f(x_1, \ldots, x_n)$ is* a sum of pairwise functions *if*

$$f(x_1, \ldots, x_n) = \sum_{A,B} f_{AB}(x_A, x_B)$$

for some functions f_{AB}.

Proposition. *When a function f is three times differentiable, then f is a sum of pairwise functions if and only if $\dfrac{\partial^3 f}{\partial x_i \partial x_j \partial x_k} = 0$ whenever i, j, and k belong to different groups.*

Proof. Let us first prove that if f is a sum of pairwise functions, then the corresponding third order derivatives are equal to 0. Without losing generality, let us assume that $i \in A$, $j \in B$, and $k \in C$. Let us first differentiate the function f with respect to x_i and x_j. The derivative of the sum is equal to the sum of the derivatives. Of all the pairwise terms forming f, only the term $f_{AB}(x_A, x_B)$ can depend both on x_i and x_j: all other terms either do not depend on x_i for $i \in A$ or do not depend on x_j for $j \in B$, and thus, the second derivatives $\dfrac{\partial^2}{\partial x_i \partial x_j}$ of all other terms are equal to 0. Thus, $\dfrac{\partial^2 f}{\partial x_i \partial x_j} = \dfrac{\partial^2 f_{AB}}{\partial x_i \partial x_j}$. The function f_{AB} depends only on the variables x_ℓ with $\ell \in A$ or $\ell \in B$. Thus, its second derivative also only depends on these variables, and cannot depend on x_k for $k \in C$ (for which $k \notin A$ and $k \notin B$). So, indeed, $\dfrac{\partial^3 f}{\partial x_i \partial x_j \partial x_k} = 0$.

Let us now prove that, vice versa, if all the corresponding third derivatives of the function $f(x_1, \ldots, x_n)$ are equal to 0, then the function $f(x_1, \ldots, x_n)$ is a sum of pairwise functions. This proof is based on the fact that if we know the partial derivative $\dfrac{\partial g}{\partial x_1}$, of a function $g(x_1, \ldots, x_n)$, then we can represent the function $g(x_1, \ldots, x_n)$ as

$$g(x_1, x_2, \ldots, x_n) = g(0, x_2, \ldots, x_n) + \int_0^{x_1} \frac{\partial g}{\partial x_1}(t, x_2, \ldots, x_n)\, dt.$$

Similarly, if we know the partial derivatives with respect to $x_1, \ldots, x_k$, then we can write

$$\begin{gathered} g(x_1, \ldots, x_k, x_{k+1}, \ldots, x_n) = g(0, \ldots, 0, x_{k+1}, \ldots, x_n) + \\ (g(x_1, 0, \ldots, 0, x_{k+1}, \ldots, x_n) - g(0, 0, \ldots, 0, x_{k+1}, \ldots, x_n)) + \\ (g(x_1, x_2, 0, \ldots, 0, x_{k+1}, \ldots, x_n) - g(x_1, 0, 0, \ldots, 0, x_{k+1}, \ldots, x_n)) + \ldots + \\ (g(x_1, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_n) - g(x_1, \ldots, x_{k-1}, 0, x_{k+1}, \ldots, x_n)) = \\ g(0, \ldots, 0, x_{k+1}, \ldots, x_n) + \int_0^{x_1} \frac{\partial g}{\partial x_1}(t, 0, \ldots, 0, x_{k+1}, \ldots, x_n)\, dt + \\ \int_0^{x_2} \frac{\partial g}{\partial x_2}(x_1, t, 0, \ldots, 0, x_{k+1}, \ldots, x_n)\, dt + \ldots + \\ \int_0^{x_k} \frac{\partial g}{\partial x_k}(x_1, x_2, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_n)\, dt. \end{gathered}$$

We have already mentioned that from the fact that f is a sum of pairwise functions, it follows that for all $i \in A$ and all $j \in B$, the partial derivative $\dfrac{\partial^2 f}{\partial x_i \partial x_j}$ depends only on the variables x_A and x_B. This second partial derivative has the form $\dfrac{\partial g}{\partial x_i}$, where $g \stackrel{\text{def}}{=} \dfrac{\partial f}{\partial x_j}$. Thus, we can get the above integral representation of the function $g = \dfrac{\partial f}{\partial x_j}$. In this representation, the first term $g(0, \ldots, 0, x_{k+1}, \ldots, x_n)$ does not depend on the variables x_A, while all other terms depend only on x_A and x_B. Thus, for every $j \in B$, we have

$$\frac{\partial f}{\partial x_j} = f_1(x_B, x_C, \ldots) + f_2(x_A, x_B)$$

for appropriate functions f_1 and f_2. Now that we have this information about the partial derivatives of the function f with respect to variables x_B, we can apply the integral formula once again and get

$$f(x_A, x_B, x_C, \ldots) = F_1(x_A, x_C, \ldots) + F_2(x_B, x_C, \ldots) + F_3(x_A, x_B)$$

for appropriate functions F_i.

When we only have three groups of variables, we have the desired representation of the function f as a sum of pairwise functions.

When we have more than three groups of variables, we can continue our decomposition. For the functions F_2 and F_3, the second order derivatives with respect to x_A and x_C are equal to 0, so $\frac{\partial^2 f}{\partial x_A \partial x_C} = \frac{\partial^2 F_1}{\partial x_A \partial x_C}$. The left-hand side depends only on x_A and x_C, thus the right-hand side also only depends on x_A and x_C. Thus, similarly to the above, we can conclude that

$$F_1(x_A, x_C, x_D, \ldots) = F_{11}(x_A, x_D, \ldots) + F_{12}(x_C, x_D, \ldots) + F_{13}(x_A, x_C).$$

A similar representation is possible for F_2, so we have

$$f(x_A, x_B, x_C, x_D, \ldots) = F_{11}(x_A, x_D, \ldots) + F_{12}(x_C, x_D, \ldots) + F_{13}(x_A, x_C) +$$

$$F_{21}(x_B, x_D, \ldots) + F_{22}(x_C, x_D, \ldots) + F_{23}(x_B, x_C) + F_3(x_A, x_B).$$

By combining F_{12} and F_{22} together into a single function F_4, we get

$$f(x_A, x_B, x_C, x_D, \ldots) = F_{11}(x_A, x_D, \ldots) + F_4(x_C, x_D, \ldots) + F_{13}(x_A, x_C) +$$

$$F_{21}(x_B, x_D, \ldots) + F_{23}(x_B, x_C) + F_3(x_A, x_B).$$

If we have four groups of variables, then the proposition is proven, otherwise we can use the same reduction once again, etc. After each reduction, we have functions depending on one fewer groups of variables, so eventually, this reduction will stop and we will get the desired representation. The proposition is proven.