Every SUE Function is a Ratio of Two Multi-Linear Functions

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Abstract

We prove that the function computed by each single-use expression is a ratio of two multi-linear functions.

1 Introduction

Importance of SUE expressions. In general, computing such a range is NP-hard [1, 2, 5], but there is a case when this range can be feasibly computed: the case of single use expressions (SUE), when each variable occurs only once [3, 4, 6].

In this case, the straightforward interval computation technique works perfectly: if we represent the computation of f as a sequence of elementary arithmetic operations and replace each operation by the corresponding interval operation, we get the exact range.

What we do in this paper. Because of the importance of SUE expressions, it is desirable to describe functions which can be computed by such expressions. In this paper, we show that all such functions are ratios of multi-linear functions.

2 Definitions and Results

Definition 1. *Let n be an integer; we will call this integer a* number of inputs*.*

- *By an* expression*, we mean a sequence of formulas of the type $s_1 := u_1 \odot_1 v_1$, $s_2 := u_2 \odot_2 v_2$, ..., $s_N := u_N \odot_N v_N$, where:*
  - *each u_i or v_i is either a rational number, or one of the inputs x_j, or one of the previous values s_k, $k < i$;*

– *each $\odot_i$ is either addition $+$, or subtraction $-$, or multiplication $\cdot$, or division $/$.*

- *By the* value *of the expression for given inputs $x_1, \ldots, x_n$, we mean the value s_N that we get after we perform all N arithmetic operations $s_i := u_i \odot_i v_i$.*

Definition 2. *An expression is called a* single use expression *(or* SUE*, for short), if each variable x_j and each term s_k appear at most once in the right-hand side of the rules $s_i := u_i \odot_i v_i$.*

Examples. An expression $1/(1 + x_2/x_1)$ corresponds to the following sequence of rules:

$$s_1 := x_2/x_1; \quad s_2 := 1 + s_1; \quad s_3 = 1/s_2.$$

One can see that in this case, each x_j and each s_k appears at most once in the right-hand side of the rules.

Definition 3. *A function $f(x_1, \ldots, x_n)$ is called* multi-linear *if it is a linear function of each variable.*

Comment. For $n = 2$, a general bilinear function has the form

$$f(x_1, x_2) = a_0 + a_1 \cdot x_1 + a_2 \cdot x_2 + a_{12} \cdot x_1 \cdot x_2.$$

A general multi-linear function has the form $f(x_1, \ldots, x_n) = \sum\limits_{I \subseteq \{1,\ldots,n\}} a_I \cdot \prod\limits_{i \in I} x_i$.

Main Result. *The function computed by each SUE expression is a ratio of two multi-linear functions.*

Auxiliary Result. *Not every multi-linear function can be computed by a SUE expression.*

Comment. As we will see from the proof, this auxiliary result remains valid if, in addition to elementary arithmetic operations, we also allow additional differential unary and binary operations (e.g., computing values of special functions of one or two variables).

3 Proofs

Proof of the Main Result. We will prove this result by induction: we will start with $n = 1$, and then we will use induction to prove this result for a general n.

1°. Let us start with the case $n = 1$. Let us prove that for SUE expressions of one variable, in each rule $s_i := u_i \odot_i v_i$, at least one of u_i and v_i is a constant.

Indeed, it is known that an expression for s_i can be naturally represented as a tree: we start with s_i as a root, and add two branches leading to u_i and v_i. If u_i or v_i is an input, we stop branching, so the input will be a leaf of the tree. If

u_i or v_i is an auxiliary quantity s_k, quantity that come from the corresponding rule $s_k := u_k \odot_k v_k$, then we add two branches leading to u_k and v_k, etc. Since each x_j or s_i can occur only once in the right-hand side, this means that all nodes of this tree are different. In particular, this means that there is only one node x_j. This node is either in the branch u_i or in the branch v_i. In both case, one of the terms u_i and v_i does not depend on x_j and is, thus, a constant.

Let us show, by (secondary) induction, that all SUE expressions with one input are fractionally linear, i.e., have the form $f(x_1) = \dfrac{a \cdot x_1 + b}{c \cdot x_1 + d}$, with rational values a, b, c, and d. Indeed:

- the variable x_1 and a constant are of this form, and
- one can easily show that as a result of an arithmetic operation between a fractional-linear function $f(x_1)$ and a constant r, we also get an expression of this form, i.e., $f(x_1) + r$, $f(x_1) - r$, $r - f(x_1)$, $r \cdot f(x_1)$, $r/f(x_1)$, and $f(x_1)/r$ are also fractionally linear.

2°. Let us now assume that we already proved his result for $n = k$, and we want to prove it for functions of $n = k+1$ variables. Since this function is SUE, we can find the first stage on which the intermediate result depends on all n variables. This means that this result comes from applying an arithmetic operation to two previous results both of which depended on fewer than n variables. Each of the two previous results thus depends on $< k + 1$ variables, i.e., on $\le k$ variables. Hence, we can conclude that each of these two previous results is a ratio of two multi-linear functions.

Since this is SUE, there two previous results depend on non-intersecting sets of variables. Without losing generality, let $x_1, \ldots, x_f$ be the variables used in the first of these previous result, and $x_{f+1}, \ldots, x_n$ are the variables used in the second of these two previous results. Then the two previous results have the form $\dfrac{N_1(x_1 \ldots, x_f)}{D_1(x_1, \ldots, x_f)}$ and $\dfrac{N_2(x_{f+1} \ldots, x_n)}{D_2(x_{f+1}, \ldots, x_n)}$, where N_i and D_i are bilinear functions. For all four arithmetic operations, we can see that the result of applying this operation is also a ratio of two multi-linear functions:

$$\frac{N_1(x_1 \ldots, x_f)}{D_1(x_1, \ldots, x_f)} + \frac{N_2(x_{f+1} \ldots, x_n)}{D_2(x_{f+1}, \ldots, x_n)} =$$

$$\frac{N_1(x_1 \ldots, x_f) \cdot D_2(x_{f+1}, \ldots, x_n) + D_1(x_1 \ldots, x_f) \cdot N_2(x_{f+1}, \ldots, x_n)}{D_1(x_1 \ldots, x_f) \cdot D_2(x_{f+1}, \ldots, x_n)};$$

$$\frac{N_1(x_1 \ldots, x_f)}{D_1(x_1, \ldots, x_f)} - \frac{N_2(x_{f+1} \ldots, x_n)}{D_2(x_{f+1}, \ldots, x_n)} =$$

$$\frac{N_1(x_1 \ldots, x_f) \cdot D_2(x_{f+1}, \ldots, x_n) - D_1(x_1 \ldots, x_f) \cdot N_2(x_{f+1}, \ldots, x_n)}{D_1(x_1 \ldots, x_f) \cdot D_2(x_{f+1}, \ldots, x_n)};$$

$$\frac{N_1(x_1 \ldots, x_f)}{D_1(x_1, \ldots, x_f)} \cdot \frac{N_2(x_{f+1} \ldots, x_n)}{D_2(x_{f+1}, \ldots, x_n)} = \frac{N_1(x_1 \ldots, x_f) \cdot N_2(x_{f+1}, \ldots, x_n)}{D_1(x_1 \ldots, x_f) \cdot D_2(x_{f+1}, \ldots, x_n)};$$

$$\left(\frac{N_1(x_1\ldots,x_f)}{D_1(x_1,\ldots,x_f)}\right):\left(\frac{N_2(x_{f+1}\ldots,x_n)}{D_2(x_{f+1},\ldots,x_n)}\right)=\frac{N_1(x_1\ldots,x_f)\cdot D_2(x_{f+1},\ldots,x_n)}{D_1(x_1\ldots,x_f)\cdot N_2(x_{f+1},\ldots,x_n)}.$$

After that, we perform arithmetic operations between a previous result and a constant – since neither of the n variables can be used again.

Similar to Part 1 of this proof, we can show that the result of an arithmetic operation between a ratio $f(x_1, x_2, \ldots, x_n)$ of two multi-linear functions and a constant r, we also get a similar ratio.

The proposition is proven.

Proof of the auxiliary result. Let us prove, by contradiction, that a bilinear function $f(x_1, x_2, x_3) = x_1 \cdot x_2 + x_2 \cdot x_3 + x_2 \cdot x_3$ cannot be represented as a SUE expression. Indeed, suppose that it is SUE. This means that first, we combine the values of two of these variables, and then we combine the result of this combination with the third of the variables. Without losing generality, we can assume that first we combine x_1 and x_2, and then add x_3 to this combination, i.e., that our function has the form $f(x_1, x_2, x_3) = F(a(x_1, x_2), x_3)$ for some functions $a(x_1, x_2)$ and $F(a, x_3)$.

The function obtained on each intermediate step is a composition of elementary (arithmetic) operations. These elementary operations are differentiable, and thus, their compositions $a(x_1, x_2)$ and $F(a, x_3)$ are also differentiable. Differentiating the above expression for f in terms of F and a by x_1 and x_2, we conclude that

$$\frac{\partial f}{\partial x_1}=\frac{\partial F}{\partial a}(a(x_1,x_2),x_3)\cdot\frac{\partial a}{\partial x_1}(x_1,x_2)$$

and

$$\frac{\partial f}{\partial x_2}=\frac{\partial F}{\partial a}(a(x_1,x_2),x_3)\cdot\frac{\partial a}{\partial x_2}(x_1,x_2).$$

Dividing the first of these equalities by the second one, we see that the terms $\dfrac{\partial F}{\partial a}$ cancel each other. Thus, the ratio of the two derivatives of f is equal to the ratio of two derivatives of a and therefore, depends only on x_1 and x_2:

$$\frac{\dfrac{\partial f}{\partial x_1}}{\dfrac{\partial f}{\partial x_2}}=\frac{\dfrac{\partial a}{\partial x_1}(x_1,x_2)}{\dfrac{\partial a}{\partial x_1}(x_1,x_2)}.$$

However, for the above function $f(x_1, x_2, x_3)$, we have $\dfrac{\partial f}{\partial x_1} = x_2 + x_3$ and $\dfrac{\partial f}{\partial x_2} = x_1 + x_3$. The ratio $\dfrac{x_2 + x_3}{x_1 + x_3}$ of these derivatives clearly depends on x_3 as well – and we showed that in the SUE case, this ratio should only depend on x_1 and x_2. The contradiction proves that this function cannot be represented in SUE form. The proposition is proven.

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