# Every Function Computable by an Arithmetic SUE Expression is a Ratio of Two Multi-Linear Functions: A Theorem

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**Abstract**

One of the main problems of interval computation is computing the range of a given function $f(x_1, \ldots, x_n)$ on a given box $[\underline{x}_1, \overline{x}_1] \times \ldots \times [\underline{x}_n, \overline{x}_n]$. In general, computing the exact range is computationally difficult (NP-hard) problem, but there are important cases when a feasible algorithm for computing such a function is possible. One of such cases is the case of singe-use expressions (SUE), when each variable occurs only once. Because of this, practitioners often try to come up with an equivalent SUE expression for computing a given function. It is therefore important to know when an expression can be transformed into a SUE form. In this paper, we consider the case of functions that can be computed by using only arithmetic operations (+, −, multiplication, and division). We show that when such a function is equivalent to a SUE, then it is equal to a ratio of two multi-linear functions (although, of course, not every such ratio can be transformed into a SUE expression). Thus, if a function cannot be represented as such a ratio, then we should not waste our efforts on finding an equivalent SUE form.

## 1 Introduction

**Importance of SUE expressions.** One of the main problems of interval computations is computing the range of a given function $f(x_1, \ldots, x_n)$ when we know the range $[\underline{x}_i, \overline{x}_i]$ of possible values of each variable $x_i$. In general, computing such a range is NP-hard [1, 2, 5], but there is a case when this range can be feasibly computed: the case of single use expressions (SUE), when each variable occurs only once [3, 4, 6].

In this case, the straightforward interval computation technique works perfectly: if we represent the computation of $f$ as a sequence of elementary arithmetic operations and replace each operation by the corresponding interval operation, we get the exact range.

Often, the original expression of a function is not SUE, but this function can be represented in an equivalent SUE form: for example, the expression $f(x_1, x_2) = \dfrac{x_1}{x_1 + x_2}$ is not SUE, but (at least for $x_1 \neq 0$) it is equivalent to a SUE expression $f(x_1, x_2) = \dfrac{1}{1 + \dfrac{x_2}{x_1}}$.

Since transformation into an equivalent SUE form helps compute the range of a function, it is important to analyze which functions can be transformed into an equivalent SUE form.

**What we do in this paper.** In this paper, we consider functions which can be computed by a finite sequence of arithmetic operations (+, −, multiplication, and division). We prove that if a function of this type can be transformed into an equivalent SUE form, then this function is a ratio of two multi-linear functions (although not all such ratios can be transformed into an equivalent SUE form).

## 2 Definitions and Results

**Definition 1.** *Let $n$ be an integer; we will call this integer a* number of inputs.

- *By an* arithmetic expression, *we mean a sequence of formulas of the type $s_1 := u_1 \odot_1 v_1$, $s_2 := u_2 \odot_2 v_2$, ..., $s_N := u_N \odot_N v_N$, where:*
  - *each $u_i$ or $v_i$ is either a rational number, or one of the inputs $x_j$, or one of the previous values $s_k$, $k < i$;*
  - *each $\odot_i$ is either addition $+$, or subtraction $-$, or multiplication $\cdot$, or division $/$.*
- *By the* value *of the expression for given inputs $x_1, \ldots, x_n$, we mean the value $s_N$ that we get after we perform all $N$ arithmetic operations $s_i := u_i \odot_i v_i$.*

**Definition 2.** *An arithmetic expression is called a* single use expression *(or* SUE, *for short), if each variable $x_j$ and each term $s_k$ appear at most once in the right-hand side of the rules $s_i := u_i \odot_i v_i$.*

**Example.** An expression $1/(1 + x_2/x_1)$ corresponds to the following sequence of rules:

$$s_1 := x_2/x_1; \quad s_2 := 1 + s_1; \quad s_3 = 1/s_2.$$

One can see that in this case, each $x_j$ and each $s_k$ appears at most once in the right-hand side of the rules.

**Definition 3.** *We say that a function $f(x_1, \ldots, x_n)$* can be computed by an arithmetic SUE expression *if there exists an arithmetic SUE expression whose value, for each tuple $(x_1, \ldots, x_n)$, is equal to $f(x_1, \ldots, x_n)$.*

**Example.** The function $f(x_1, x_2) = \dfrac{x_1}{x_1 + x_2}$ is not itself SUE, but it can be computed by the above SUE expression $1/(1 + x_2/x_1)$.

**Definition 4.** *A function $f(x_1, \ldots, x_n)$ is called* multi-linear *if it is a linear function of each variable.*

*Comment.* For $n = 2$, a general bilinear function has the form

$$f(x_1, x_2) = a_0 + a_1 \cdot x_1 + a_2 \cdot x_2 + a_{12} \cdot x_1 \cdot x_2.$$

A general multi-linear function has the form $f(x_1, \ldots, x_n) = \sum\limits_{I \subseteq \{1,\ldots,n\}} a_I \cdot \prod\limits_{i \in I} x_i$.

**Main Result.** *If a function can be computed by an arithmetic SUE expression, then this function is equal to a ratio of two multi-linear functions.*

**Auxiliary Result.** *Not every multi-linear function can be computed by an arithmetic SUE expression.*

*Comment.* As we will see from the proof, this auxiliary result remains valid if, in our definition of a SUE expression, in addition to elementary arithmetic operations, we also allow additional differential unary and binary operations (e.g., computing values of special functions of one or two variables).

# 3 Proofs

**Proof of the Main Result.** The main result means, in effect, that for each arithmetic SUE expression, the corresponding function $f(x_1, \ldots, x_n)$ is equal to a ratio of two multi-linear functions. We will prove this result by induction: we will start with $n = 1$, and then we will use induction to prove this result for a general $n$.

$1^\circ$. Let us start with the case $n = 1$. Let us prove that for arithmetic SUE expressions of one variable, in each rule $s_i := u_i \odot_i v_i$, at least one of $u_i$ and $v_i$ is a constant.

Indeed, it is known that an expression for $s_i$ can be naturally represented as a tree: we start with $s_i$ as a root, and add two branches leading to $u_i$ and $v_i$. If $u_i$ or $v_i$ is an input, we stop branching, so the input will be a leaf of the tree. If $u_i$ or $v_i$ is an auxiliary quantity $s_k$, quantity that come from the corresponding rule $s_k := u_k \odot_k v_k$, then we add two branches leading to $u_k$ and $v_k$, etc. Since each $x_j$ or $s_i$ can occur only once in the right-hand side, this means that all nodes of this tree are different. In particular, this means that there is only one node $x_j$. This node is either in the branch $u_i$ or in the branch $v_i$. In both case, one of the terms $u_i$ and $v_i$ does not depend on $x_j$ and is, thus, a constant.

Let us show, by (secondary) induction, that all arithmetic SUE expressions with one input are fractionally linear, i.e., have the form $f(x_1) = \dfrac{a \cdot x_1 + b}{c \cdot x_1 + d}$, with rational values $a$, $b$, $c$, and $d$. Indeed:

- the variable $x_1$ and a constant are of this form, and
- one can easily show that as a result of an arithmetic operation between a fractional-linear function $f(x_1)$ and a constant $r$, we also get an expression of this form, i.e., $f(x_1) + r$, $f(x_1) - r$, $r - f(x_1)$, $r \cdot f(x_1)$, $r/f(x_1)$, and $f(x_1)/r$ are also fractionally linear.

*Comment.* It is worth mentioning that, vice versa, each fractionally linear function $f(x_1) = \dfrac{a \cdot x_1 + b}{c \cdot x_1 + d}$ can be computed by an arithmetic SUE expression. Indeed, if $c = 0$, then $f(x_1)$ is a linear function $f(x_1) = \dfrac{a}{d} \cdot x_1 + \dfrac{b}{d}$, and is, thus, clearly SUE.

When $c \neq 0$, then this function can be represented in the following equivalent SUE form: $f(x_1) = \dfrac{a}{c} + \dfrac{b - \dfrac{a \cdot d}{c}}{c \cdot x_1 + d}$.

2°. Let us now assume that we already proved his result for $n = k$, and we want to prove it for functions of $n = k + 1$ variables. Since this function can be computed by an arithmetic SUE expression, we can find the first stage on which the intermediate result depends on all $n$ variables. This means that this result comes from applying an arithmetic operation to two previous results both of which depended on fewer than $n$ variables. Each of the two previous results thus depends on $< k+1$ variables, i.e., on $\leq k$ variables. Hence, we can conclude that each of these two previous results is a ratio of two multi-linear functions.

Since this is SUE, there two previous results depend on non-intersecting sets of variables. Without losing generality, let $x_1, \ldots, x_f$ be the variables used in the first of these previous result, and $x_{f+1}, \ldots, x_n$ are the variables used in the second of these two previous results. Then the two previous results have the form $\dfrac{N_1(x_1 \ldots, x_f)}{D_1(x_1, \ldots, x_f)}$ and $\dfrac{N_2(x_{f+1} \ldots, x_n)}{D_2(x_{f+1}, \ldots, x_n)}$, where $N_i$ and $D_i$ are bilinear functions. For all four arithmetic operations, we can see that the result of applying this operation is also a ratio of two multi-linear functions:

$$\frac{N_1(x_1 \ldots, x_f)}{D_1(x_1, \ldots, x_f)} + \frac{N_2(x_{f+1} \ldots, x_n)}{D_2(x_{f+1}, \ldots, x_n)} =$$

$$\frac{N_1(x_1 \ldots, x_f) \cdot D_2(x_{f+1}, \ldots, x_n) + D_1(x_1 \ldots, x_f) \cdot N_2(x_{f+1}, \ldots, x_n)}{D_1(x_1 \ldots, x_f) \cdot D_2(x_{f+1}, \ldots, x_n)};$$

$$\frac{N_1(x_1 \ldots, x_f)}{D_1(x_1, \ldots, x_f)} - \frac{N_2(x_{f+1} \ldots, x_n)}{D_2(x_{f+1}, \ldots, x_n)} =$$

$$\frac{N_1(x_1\ldots,x_f)\cdot D_2(x_{f+1},\ldots,x_n)-D_1(x_1\ldots,x_f)\cdot N_2(x_{f+1},\ldots,x_n)}{D_1(x_1\ldots,x_f)\cdot D_2(x_{f+1},\ldots,x_n)};$$

$$\frac{N_1(x_1\ldots,x_f)}{D_1(x_1,\ldots,x_f)}\cdot\frac{N_2(x_{f+1}\ldots,x_n)}{D_2(x_{f+1},\ldots,x_n)}=\frac{N_1(x_1\ldots,x_f)\cdot N_2(x_{f+1},\ldots,x_n)}{D_1(x_1\ldots,x_f)\cdot D_2(x_{f+1},\ldots,x_n)};$$

$$\left(\frac{N_1(x_1\ldots,x_f)}{D_1(x_1,\ldots,x_f)}\right):\left(\frac{N_2(x_{f+1}\ldots,x_n)}{D_2(x_{f+1},\ldots,x_n)}\right)=\frac{N_1(x_1\ldots,x_f)\cdot D_2(x_{f+1},\ldots,x_n)}{D_1(x_1\ldots,x_f)\cdot N_2(x_{f+1},\ldots,x_n)}.$$

After that, we perform arithmetic operations between a previous result and a constant – since neither of the $n$ variables can be used again.

Similar to Part 1 of this proof, we can show that the result of an arithmetic operation between a ratio $f(x_1,x_2,\ldots,x_n)$ of two multi-linear functions and a constant $r$, we also get a similar ratio.

The proposition is proven.

**Proof of the auxiliary result.** Let us prove, by contradiction, that a bilinear function $f(x_1,x_2,x_3)=x_1\cdot x_2+x_2\cdot x_3+x_2\cdot x_3$ cannot be computed by a SUE expression. Indeed, suppose that there is a SUE expression that computes this function. By definition of SUE, this means that first, we combine the values of two of these variables, and then we combine the result of this combination with the third of the variables. Without losing generality, we can assume that first we combine $x_1$ and $x_2$, and then add $x_3$ to this combination, i.e., that our function has the form $f(x_1,x_2,x_3)=F(a(x_1,x_2),x_3)$ for some functions $a(x_1,x_2)$ and $F(a,x_3)$.

The function obtained on each intermediate step is a composition of elementary (arithmetic) operations. These elementary operations are differentiable, and thus, their compositions $a(x_1,x_2)$ and $F(a,x_3)$ are also differentiable. Differentiating the above expression for $f$ in terms of $F$ and $a$ by $x_1$ and $x_2$, we conclude that

$$\frac{\partial f}{\partial x_1}=\frac{\partial F}{\partial a}(a(x_1,x_2),x_3)\cdot\frac{\partial a}{\partial x_1}(x_1,x_2)$$

and

$$\frac{\partial f}{\partial x_2}=\frac{\partial F}{\partial a}(a(x_1,x_2),x_3)\cdot\frac{\partial a}{\partial x_2}(x_1,x_2).$$

Dividing the first of these equalities by the second one, we see that the terms $\frac{\partial F}{\partial a}$ cancel each other. Thus, the ratio of the two derivatives of $f$ is equal to the ratio of two derivatives of $a$ and therefore, depends only on $x_1$ and $x_2$:

$$\frac{\dfrac{\partial f}{\partial x_1}}{\dfrac{\partial f}{\partial x_2}}=\frac{\dfrac{\partial a}{\partial x_1}(x_1,x_2)}{\dfrac{\partial a}{\partial x_1}(x_1,x_2)}.$$

However, for the above function $f(x_1,x_2,x_3)$, we have $\frac{\partial f}{\partial x_1}=x_2+x_3$ and $\frac{\partial f}{\partial x_2}=x_1+x_3$. The ratio $\frac{x_2+x_3}{x_1+x_3}$ of these derivatives clearly depends on $x_3$ as well – and we showed that in the SUE case, this ratio should only depend on $x_1$ and $x_2$. The contradiction proves that this function cannot be computed by a SUE expression. The proposition is proven.

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## References

[1] A. A. Gaganov, *Computational complexity of the range of the polynomial in several variables*, Leningrad University, Math. Department, M.S. Thesis, 1981 (in Russian).

[2] A. A. Gaganov, "Computational complexity of the range of the polynomial in several variables", *Cybernetics*, 1985, pp. 418–421.

[3] E. Hansen, "Sharpness in interval computations", *Reliable Computing*, 1997, Vol. 3, pp. 7–29.

[4] L. Jaulin, M. Kieffer, O. Didrit, and E. Walter, *Applied Interval Analysis, with Examples in Parameter and State Estimation, Robust Control and Robotics*, Springer-Verlag, London, 2001.

[5] V. Kreinovich, A. Lakeyev, J. Rohn, and P. Kahl, *Computational complexity and feasibility of data processing and interval computations*, Kluwer, Dordrecht, 1998.

[6] R. E. Moore, R. B. Kearfott, and M. J. Cloud, *Introduction to Interval Analysis*, SIAM Press, Philadelphia, Pennsylviania, 2009.