Journal of Uncertain Systems
Vol.10, No.x, pp.xx-xx, 2016
Online at: www.jus.org.uk

Every Function Computable by an Arithmetic Single Use Expression Is a Ratio of Two Multi-Linear Functions: A Theorem

Joe Lorkowski, Olga Kosheleva, and Vladik Kreinovich
University of Texas at El Paso
El Paso, TX 79968, USA
lorkowski@computer.org, olgak@utep.edu, vladik@utep.edu

Received 11 February 2015; Revised 27 April 2015

Abstract

One of the main problems of interval computation is computing the range of a given function on a given box. In general, computing the exact range is a computationally difficult (NP-hard) problem, but there are important cases when a feasible algorithm for computing such a function is possible. One of such cases is the case of singe-use expressions (SUE), when each variable occurs only once. Because of this, practitioners often try to come up with a SUE expression for computing a given function. It is therefore important to know when such a SUE expression is possible. In this paper, we consider the case of functions that can be computed by using only arithmetic operations (addition, subtraction, multiplication, and division). We show that when there exists a SUE expression for computing such a function, then this function is equal to a ratio of two multi-linear functions (although there are ratios of multi-linear functions for which no SUE expression is possible). Thus, if a function is not equal to such a ratio, then we should not waste our efforts on finding a SUE expression for computing this function.

Keywords: interval computations, single use expressions, uncertainty propagation

1 Introduction

Importance of interval computations. In many practical situations, we are interested in the value of some quantity y which is difficult or even impossible to measure directly. This may be a difficult-to-directly-measure quantity, such as the distance to a star or the amount of oil in an oil well; this may be a future value of a quantity.

To estimate such values y, we find easier-to-measure quantities $x_1, \ldots, x_n$ that are related to y by a known dependence $y = f(x_1, \ldots, x_n)$, measure these quantities x_i, and use the results $\widetilde{x}_i$ of these measurements to compute the estimate $\widetilde{y} = f(\widetilde{x}_1, \ldots, \widetilde{x}_n)$ for the desired quantity y.

Measurements are never absolutely accurate. As a result, the actual value x_i of the corresponding quantities is, in general, different from the measurement results $\widetilde{x}_i$. Because of this difference, the estimate $\widetilde{y}$ based on measurement results is, in general, somewhat different from the actual value $y = f(x_1, \ldots, x_n)$. In practical applications, it is important to know how accurate it is the estimate $\widetilde{x}_i$.

In many practical situations, the only information that we have about the measurement error $\Delta x_i \stackrel{\text{def}}{=} \widetilde{x}_i - x_i$ is the upper bound Δ_i on its absolute value: $|\Delta x_i| \le \Delta_i$. In such situations, after each measurement, the only information that we have about the actual (unknown) value x_i is that it belongs to the interval $[\underline{x}_i, \overline{x}_i] = [\widetilde{x}_i - \Delta_i, \widetilde{x}_i + \Delta_i]$.

Different combinations $(x_1, \ldots, x_n)$ of possible values $x_i \in [\underline{x}_i, \overline{x}_i]$ lead, in general, to different values of $y = f(x_1, \ldots, x_n)$. Our task is thus to find the range of all possible values of y, i.e., the interval

$$\{f(x_1, \ldots, x_n) : x_1 \in [\underline{x}_1, \overline{x}_1], \ldots, x_n \in [\underline{x}_n, \overline{x}_n]\}.$$

The problem of computing this range is known as the problem of *interval computations*; see, e.g., [5, 8].

Importance of SUE expressions. In general, computing the desired range is NP-hard [1, 2, 6]. This means, in particular, that if P≠NP (as most computer scientists believe), then it is not possible to have a feasible (= polynomial-time) algorithm that would always compute the desired range.

There are, however, cases when this range can be feasibly computed. One of such cases is when each variable occurs only once. The fact that in this case, we get the exact range was first mentioned by R. E.

Moore himself in his 1966 book [7]; see also [3, 4, 5, 8, 9]. Following Bill Walster (see, e.g., [4, 9]), we will call such expressions *single use expressions* (SUE, for short)

In this case, the straightforward interval computation technique works perfectly: if we represent the computation of f as a sequence of elementary arithmetic operations and replace each operation by the corresponding interval operation, then – modulo rounding errors – we get the exact range.

Often, the original expression of a function is not SUE, but there is a SUE expression for computing the same function: for example, the expression $f(x_1,x_2)=\dfrac{x_1}{x_1+x_2}$ is not SUE, but (at least for $x_1\neq 0$) this function can be computed by a SUE expression $f(x_1,x_2)=\dfrac{1}{1+\dfrac{x_2}{x_1}}$.

Since a SUE expression helps to compute the range of a function, it is important to analyze for which functions such SUE expressions are possible.

Comment. To be more precise, if we only use real numbers, then the above two expressions have different domains:

- the first expression $f(x_1,x_2)=\dfrac{x_1}{x_1+x_2}$ is well-defined when $x_1=0$ and $x_2\neq 0$ (and equal to 0 in this case), while
- the second expression is not defined when $f(x_1,x_2)=\dfrac{1}{1+\dfrac{x_2}{x_1}}$.

Most computers, however, follow the IEEE Standard 754 when performing computations on real numbers, and this standard requires that the values $+\infty$ and $-\infty$ are also legitimate computer-represented real numbers. In this case, the second expression is also defined for $x_1=0$ and $x_2\neq 0$: namely, we get $\dfrac{x_2}{x_1}=\infty$, thus, $1+\dfrac{x_2}{x_1}=1+\infty=\infty$, and $f(x_1,x_2)=\dfrac{1}{1+\dfrac{x_2}{x_1}}=\dfrac{1}{\infty}=0$.

This is one of the examples that motivated the introduction of infinities as possible values into the IEEE Standard 754.

What we do in this paper. In this paper, we consider functions which can be computed by a finite sequence of arithmetic operations ($+$, $-$, multiplication, and division). We prove that if for such a function, there is a SUE expression computing this function, then this function is a ratio of two multi-linear functions (although there are ratios of multi-linear functions for which no SUE expression is possible).

2 Definitions and Results

Definition 1. *Let n be an integer; we will call this integer a* number of inputs*.*

- *By an* arithmetic expression*, we mean a sequence of formulas of the type $s_1:=u_1\odot_1 v_1$, $s_2:=u_2\odot_2 v_2$, …, $s_N:=u_N\odot_N v_N$, where:*
  - *each u_i or v_i is either a rational number, or one of the inputs x_j, or one of the previous values s_k, $k<i$;*
  - *each $\odot_i$ is either addition $+$, or subtraction $-$, or multiplication $\cdot$, or division $/$.*
- *By the* value *of the expression for given inputs $x_1,\ldots,x_n$, we mean the value s_N that we get after we perform all N arithmetic operations $s_i:=u_i\odot_i v_i$.*

Definition 2. *An arithmetic expression is called a* single use expression *(or* SUE*, for short), if each variable x_j and each term s_k appear at most once in the right-hand side of the rules $s_i:=u_i\odot_i v_i$.*

Example. An expression $1/(1+x_2/x_1)$ corresponds to the following sequence of rules:

$$s_1:=x_2/x_1;\quad s_2:=1+s_1;\quad s_3=1/s_2.$$

One can see that in this case, each x_j and each s_k appears at most once in the right-hand side of the rules.

Definition 3. *We say that a function $f(x_1,\ldots,x_n)$ can be computed by an* arithmetic SUE expression *if there exists an arithmetic SUE expression whose value, for each tuple $(x_1,\ldots,x_n)$, is equal to $f(x_1,\ldots,x_n)$.*

Example. The function $f(x_1,x_2)=\dfrac{x_1}{x_1+x_2}$ is not itself SUE, but it can be computed by the above SUE expression $1/(1+x_2/x_1)$.

Definition 4. *A function $f(x_1,\ldots,x_n)$ is called* multi-linear *if it is a linear function of each variable.*

Comment. For $n=2$, a general bilinear function has the form

$$f(x_1,x_2)=a_0+a_1\cdot x_1+a_2\cdot x_2+a_{1,2}\cdot x_1\cdot x_2.$$

A general multi-linear function has the form $f(x_1,\ldots,x_n)=\sum\limits_{I\subseteq\{1,\ldots,n\}} a_I\cdot\prod\limits_{i\in I} x_i$.

For example, if we take $I=\{1,2\}$, then:

- for $I=\emptyset$, we get the free term a_0;
- for $I=\{1\}$, we get the term $a_1\cdot x_1$;
- for $I=\{2\}$, we get the term $a_2\cdot x_2$, and
- for $I=\{1,2\}$, we get the term $a_{1,2}\cdot x_1\cdot x_2$.

Main Result. *If a function can be computed by an arithmetic SUE expression, then this function is equal to a ratio of two multi-linear functions.*

Auxiliary Result. *Not every multi-linear function can be computed by an arithmetic SUE expression.*

Comment. As we will see from the proof, this auxiliary result remains valid if, in our definition of a SUE expression, in addition to elementary arithmetic operations, we also allow additional differential unary and binary operations (e.g., computing values of special functions of one or two variables).

3 Proofs

Proof of the Main Result. The main result means, in effect, that for each arithmetic SUE expression, the corresponding function $f(x_1,\ldots,x_n)$ is equal to a ratio of two multi-linear functions. We will prove this result by induction: we will start with $n=1$, and then we will use induction to prove this result for a general n.

1°. Let us start with the case $n=1$. Let us prove that for arithmetic SUE expressions of one variable, in each rule $s_i:=u_i\odot_i v_i$, at least one of u_i and v_i is a constant.

Indeed, it is known that an expression for s_i can be naturally represented as a tree:

- We start with s_i as a root, and add two branches leading to u_i and v_i.
- If u_i or v_i is an input, we stop branching, so the input will be a leaf of the tree.
- If u_i or v_i is an auxiliary quantity s_k, quantity that come from the corresponding rule $s_k:=u_k\odot_k v_k$, then we add two branches leading to u_k and v_k, etc.

Since each x_j or s_i can occur only once in the right-hand side, this means that all nodes of this tree are different. In particular, this means that there is only one node x_j. This node is either in the branch u_i or in the branch v_i. In both case, one of the terms u_i and v_i does not depend on x_j and is, thus, a constant.

Let us show, by (secondary) induction, that all arithmetic SUE expressions with one input are fractionally linear, i.e., have the form $f(x_1)=\dfrac{a\cdot x_1+b}{c\cdot x_1+d}$, with rational values a, b, c, and d. Indeed:

- the variable x_1 and a constant are of this form, and

- one can easily show that as a result of an arithmetic operation between a fractional-linear function $f(x_1)$ and a constant r, we also get an expression of this form, i.e., $f(x_1)+r$, $f(x_1)-r$, $r-f(x_1)$, $r\cdot f(x_1)$, $r/f(x_1)$, and $f(x_1)/r$ are also fractionally linear.

Comment. It is worth mentioning that, vice versa, each fractionally linear function $f(x_1)=\dfrac{a\cdot x_1+b}{c\cdot x_1+d}$ can be computed by an arithmetic SUE expression. Indeed, if $c=0$, then $f(x_1)$ is a linear function $f(x_1)=\dfrac{a}{d}\cdot x_1+\dfrac{b}{d}$, and is, thus, clearly SUE.

When $c\neq 0$, then this function can be computed by using the following SUE form: $f(x_1)=\dfrac{a}{c}+\dfrac{b-\dfrac{a\cdot d}{c}}{c\cdot x_1+d}$.

2°. Let us now assume that we already proved his result for $n=k$, and we want to prove it for functions of $n=k+1$ variables. Since this function can be computed by an arithmetic SUE expression, we can find the first stage on which the intermediate result depends on all n variables. This means that this result comes from applying an arithmetic operation to two previous results both of which depended on fewer than n variables. Each of the two previous results thus depends on $<k+1$ variables, i.e., on $\le k$ variables. Hence, we can conclude that each of these two previous results is a ratio of two multi-linear functions.

Since this is SUE, there two previous results depend on non-intersecting sets of variables. Without losing generality, let $x_1,\ldots,x_f$ be the variables used in the first of these previous result, and $x_{f+1},\ldots,x_n$ are the variables used in the second of these two previous results. Then the two previous results have the form $\dfrac{N_1(x_1\ldots,x_f)}{D_1(x_1,\ldots,x_f)}$ and $\dfrac{N_2(x_{f+1}\ldots,x_n)}{D_2(x_{f+1},\ldots,x_n)}$, where N_i and D_i are bilinear functions. For all four arithmetic operations, we can see that the result of applying this operation is also a ratio of two multi-linear functions:

$$\frac{N_1(x_1\ldots,x_f)}{D_1(x_1,\ldots,x_f)}+\frac{N_2(x_{f+1}\ldots,x_n)}{D_2(x_{f+1},\ldots,x_n)}=$$

$$\frac{N_1(x_1\ldots,x_f)\cdot D_2(x_{f+1},\ldots,x_n)+D_1(x_1\ldots,x_f)\cdot N_2(x_{f+1},\ldots,x_n)}{D_1(x_1\ldots,x_f)\cdot D_2(x_{f+1},\ldots,x_n)};$$

$$\frac{N_1(x_1\ldots,x_f)}{D_1(x_1,\ldots,x_f)}-\frac{N_2(x_{f+1}\ldots,x_n)}{D_2(x_{f+1},\ldots,x_n)}=$$

$$\frac{N_1(x_1\ldots,x_f)\cdot D_2(x_{f+1},\ldots,x_n)-D_1(x_1\ldots,x_f)\cdot N_2(x_{f+1},\ldots,x_n)}{D_1(x_1\ldots,x_f)\cdot D_2(x_{f+1},\ldots,x_n)};$$

$$\frac{N_1(x_1\ldots,x_f)}{D_1(x_1,\ldots,x_f)}\cdot\frac{N_2(x_{f+1}\ldots,x_n)}{D_2(x_{f+1},\ldots,x_n)}=\frac{N_1(x_1\ldots,x_f)\cdot N_2(x_{f+1},\ldots,x_n)}{D_1(x_1\ldots,x_f)\cdot D_2(x_{f+1},\ldots,x_n)};$$

$$\left(\frac{N_1(x_1\ldots,x_f)}{D_1(x_1,\ldots,x_f)}\right):\left(\frac{N_2(x_{f+1}\ldots,x_n)}{D_2(x_{f+1},\ldots,x_n)}\right)=\frac{N_1(x_1\ldots,x_f)\cdot D_2(x_{f+1},\ldots,x_n)}{D_1(x_1\ldots,x_f)\cdot N_2(x_{f+1},\ldots,x_n)}.$$

After that, we perform arithmetic operations between a previous result and a constant – since neither of the n variables can be used again.

Similar to Part 1 of this proof, we can show that the result of an arithmetic operation between a ratio $f(x_1,x_2,\ldots,x_n)$ of two multi-linear functions and a constant r, we also get a similar ratio.

The proposition is proven.

Proof of the auxiliary result. Let us prove, by contradiction, that a bilinear function $f(x_1,x_2,x_3)=x_1\cdot x_2+x_2\cdot x_3+x_2\cdot x_3$ cannot be computed by a SUE expression. Indeed, suppose that there is a SUE expression that computes this function. By definition of SUE, this means that first, we combine the values of two of these variables, and then we combine the result of this combination with the third of the variables. Without losing generality, we can assume that first we combine x_1 and x_2, and then add x_3 to this combination, i.e., that our function has the form $f(x_1,x_2,x_3)=F(a(x_1,x_2),x_3)$ for some functions $a(x_1,x_2)$ and $F(a,x_3)$.

The function obtained on each intermediate step is a composition of elementary (arithmetic) operations. These elementary operations are differentiable, and thus, their compositions $a(x_1, x_2)$ and $F(a, x_3)$ are also differentiable. Differentiating the above expression for f in terms of F and a by x_1 and x_2, we conclude that

$$\frac{\partial f}{\partial x_1} = \frac{\partial F}{\partial a}(a(x_1, x_2), x_3) \cdot \frac{\partial a}{\partial x_1}(x_1, x_2)$$

and

$$\frac{\partial f}{\partial x_2} = \frac{\partial F}{\partial a}(a(x_1, x_2), x_3) \cdot \frac{\partial a}{\partial x_2}(x_1, x_2).$$

Dividing the first of these equalities by the second one, we see that the terms $\dfrac{\partial F}{\partial a}$ cancel each other. Thus, the ratio of the two derivatives of f is equal to the ratio of two derivatives of a and therefore, depends only on x_1 and x_2:

$$\frac{\dfrac{\partial f}{\partial x_1}}{\dfrac{\partial f}{\partial x_2}} = \frac{\dfrac{\partial a}{\partial x_1}(x_1, x_2)}{\dfrac{\partial a}{\partial x_1}(x_1, x_2)}.$$

However, for the above function $f(x_1, x_2, x_3)$, we have $\dfrac{\partial f}{\partial x_1} = x_2 + x_3$ and $\dfrac{\partial f}{\partial x_2} = x_1 + x_3$. The ratio $\dfrac{x_2 + x_3}{x_1 + x_3}$ of these derivatives clearly depends on x_3 as well – and we showed that in the SUE case, this ratio should only depend on x_1 and x_2. The contradiction proves that this function cannot be computed by a SUE expression. The proposition is proven.

Acknowledgments. This work was supported in part by the National Science Foundation grants HRD-0734825 and HRD-1242122 (Cyber-ShARE Center of Excellence) and DUE-0926721.

The authors are thankful to R. Baker Kearfott for valuable suggestions, and to the anonymous referees for useful comments.

References

[1] A. A. Gaganov, *Computational complexity of the range of the polynomial in several variables*, Leningrad University, Math. Department, M.S. Thesis, 1981 (in Russian).

[2] A. A. Gaganov, “Computational complexity of the range of the polynomial in several variables”, *Cybernetics*, 1985, pp. 418–421.

[3] E. Hansen, “Sharpness in interval computations”, *Reliable Computing*, 1997, Vol. 3, pp. 7–29.

[4] E. Hansen and G. W. Walster, *Global Optimization Using Interval Analysis*, Marcel Decker, 2004.

[5] L. Jaulin, M. Kieffer, O. Didrit, and E. Walter, *Applied Interval Analysis, with Examples in Parameter and State Estimation, Robust Control and Robotics*, Springer-Verlag, London, 2001.

[6] V. Kreinovich, A. Lakeyev, J. Rohn, and P. Kahl, *Computational complexity and feasibility of data processing and interval computations*, Kluwer, Dordrecht, 1998.

[7] R. E. Moore, *Interval Analysis*, Prentice-Hall, Englewood Cliffs, New Jersey, 1966.

[8] R. E. Moore, R. B. Kearfott, and M. J. Cloud, *Introduction to Interval Analysis*, SIAM Press, Philadelphia, Pennsylviania, 2009.

[9] G. W. Walster, “Moore’s Single-Use-Expression Theorem on extended real intervals”, *Abstracts of the 2002 SIAM Workshop on Validated Computing*, Toronto, Canada, May 23–25, 2002.