Why Are Vine Copulas So Successful in Econometrics?

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Abstract

One of the most empirically successful tools for studying dependence between different quantities in econometrics is the tool of vine copulas. In this paper, we explain this empirical success by showing that the most widely used vine copulas are, in effect, the results of using the general fuzzy methodology. To be more precise, vine copulas correspond to a natural extension of the traditional fuzzy methodology, when we allow several different "and"-operations (t-norms), and some of these t-norms can be non-associative.

1 Vine Copulas Are Successful in Econometrics: A Brief Introduction

In econometrics, it is important to study relations between several random variables. Econometrics is a name for quantitative study of economic phenomena. Economic phenomena are very complex to study, because economics involves many inter-related phenomena: inter-related countries, inter-related sectors within each country's economy, inter-related companies within each sector, etc. To properly describe economic processes, it is therefore necessary to properly describe the relation between different economics-related quantities.

How to describe relation between several random variables: general

idea. A standard way to describe the relation between several random variables $X_1, \ldots, X_n$ is by describing their joint distribution function

$$F(x_1, \ldots, x_n) \stackrel{\text{def}}{=} \text{Prob}(X_1 \le x_1 \,\&\, \ldots \,\&\, X_n \le x_n).$$

Need for approximations. To describe a generic function, a natural way is to describe its values on a grid. If we have p possible values for each variable x_i, then we have p^n tuples $(x_1, \ldots, x_n)$ describing different grid points.

For large n, the number p^n becomes astronomical, even when p is small; it easily becomes larger than number of particles in the Universe. So, it is not possible to determine all such values – and thus, it is not possible to consider generic functions of n variables. We therefore need to approximate such dependence by using functions of fewer variables.

Vine copulas: an empirically successful idea. One of the ways to approximate a distribution is known as *vine copulas* [1, 2, 3, 8, 12, 13, 14, 17, 18, 23].

This representation is based on the fact that for the case of two variables, there exists a function $C(u, v)$ for which

$$F(x_1, x_2) = C(F_1(x_1), F_2(x_2)),$$

where $F_i(x_i) \stackrel{\text{def}}{=} \text{Prob}(X_i \le x_i)$ are *marginal distributions*; this function $C(u, v)$ is known as a *copula*; see, e.g., [21, 24].

If we have the third variable X_3, then, for each value x_3 of this variable, we can have a similar relation for distributions restricted to this particular value of x_3:

$$F(x_1, x_2 \,|\, x_3) = C(F_1(x_1 \,|\, x_3), F_2(x_2 \,|\, x_3), x_3).$$

In general, for different values x_3, we get different copulas; this is why we added x_3 as an explicit parameter of the copula.

However, in the first approximation, it makes sense to ignore this dependence and to assume that the same copula $C(u, v)$ can be used for all possible values of x_3:

$$F(x_1, x_2 \,|\, x_3) = C(F_1(x_1 \,|\, x_3), F_2(x_2 \,|\, x_3)).$$

Under this assumption, to describe a joint probability distribution of three variables, it is sufficient to have three functions of two variables: $F_1(x_1 \,|\, x_3)$, $F_2(x_2 \,|\, x_3)$, and $C(u, v)$.

Similarly, we are considering one more variable, we can get an approximate representation of joint distributions of four variables, etc. Such approximate representations are known as *vine copulas* for the following reason: we reduce a single multi-dimensional representation to a sequence of dependencies between pairs of variables, this sequence of dependencies can be represented as a graph, and the resulting graph looks like a grapevine.

Such approximations have indeed been very successful in describing econometric phenomena; see, e.g., [11].

How exactly vine copulas are used in econometrics: a typical example. For each of the econometric dynamic variables $r_t \stackrel{\text{def}}{=} X_i(t)$, there are known ways to describe its dynamics. One of the most (and probably *the* most) adequate models for such a dynamics is described by an appropriate combination of the Auto-Regressive Moving-Average Model (ARMA) and the Glosten-Jagannathan-Runkle (GJR) form of a Generalized Auto-Regressive Conditional Heteroskedasticity (GARCH) model [4]; see, e.g., [7, 19]. The corresponding ARMA(p,q)-GJR(k,ℓ) model has the form

$$r_t = c + \sum_{i=1}^{p} \varphi_i \cdot r_{t-i} + \varepsilon_i \sum_{j=1}^{q} \psi_j \cdot \varepsilon_{t-j},$$

$$\varepsilon_t = h_t \cdot \eta_t,$$

$$h_t^2 = \omega + \sum_{i=1}^{k} \alpha_i \cdot \varepsilon_{t-i}^2 + \sum_{i:\ \varepsilon_{t-i}<0} \gamma_i \cdot \varepsilon_{t-i}^2 + \sum_{j=1}^{\ell} \beta_j \cdot h_{t-j}^2,$$

where ε_t and h_t are auxiliary variables, c, φ_i, ψ_j, ω, α_i, and β_j are real-valued constants (which need to be determined based on the observations), and residuals η_t corresponding to different moments of time t are independent identically distributed random variables.

The distribution of the residuals is usually assumed to be distributed according to skewed student-t or skewed Generalized Error Distribution (GED). A skewed t-distribution means that we combine, with fixed weights, t-distributions $f_1(x)$ and $f_2(x)$ with different scalar parameters limited to, correspondingly, positive and negative values x_i: $f(x) = w_1 \cdot f_1(x)$ when $x \ge 0$ and $f(x) = w_2 \cdot f_2(x)$ when $x < 0$.

A GED distribution is a distribution with a probability density proportional to $\exp\left(-\frac{|x|^\nu}{\sigma^\nu}\right)$; it generalizes Gaussian distribution – which corresponds to $\nu = 2$. A skewed GED distribution is a combination of two GED distributions $f_1(x)$ and $f_2(x)$ corresponding to different values σ (but the same value ν): $f(x) = w_1 \cdot f_1(x)$ when $x \ge 0$ and $f(x) = w_2 \cdot f_2(x)$ when $x < 0$, where w_i are appropriate weights.

Vine copulas are used to describe, at each moment of time t, the dependence between the residuals η_t corresponding to different variables; see, e.g., [20] (the *residuals* η_t and $\eta_{t'}$ corresponding to different moments of time are usually assumed to be independent).

Problem. The use of vine copulas is a semi-heuristic idea: it makes sense, but one can think of many other possible ways to approximate a function of several variables by functions of fewer variables. So why is namely vine copula approximation so successful in econometrics?

What we do in this paper. In this paper, we explain the empirical success by vine copulas by showing that the most widely used vine copulas are, in effect, the results of using the general fuzzy methodology. To be more precise, vine copulas correspond to a natural extension of the traditional fuzzy methodology, when we allow several different "and"-operations (t-norms), and some of these t-norms can be non-associative.

Comment. Some ideas from this paper further develop results and suggestions presented in [25].

2 Fuzzy Methodology: A Brief Overview

In order to explain how fuzzy methodology is related to vine copulas, we need to recall where fuzzy methodology comes from.

The origin of fuzzy methodology: the need to describe expert knowledge. Fuzzy methodology [15, 22, 27] originated from the need to describe expert knowledge in precise computer-understandable terms.

In many situations, we do not have an exact algorithm for solving the corresponding class of problems: there is still no algorithm for driving a car, there is no algorithm for curing diseases, etc. In such situations, we relay on experts.

Some of these experts are much better than others. It is therefore desirable to use the knowledge of the best experts to help other experts make the corresponding decisions.

How to represent expert knowledge: ideal case. The ultimate goal of an expert is to recommend the appropriate action y based on the values of the corresponding variables $x_1, \ldots, x_n$. For example, for a car, we want to make a decision on breaking, turning, etc. (y) based on the values $x_1, \ldots, x_n$ that describe location and velocity of this car and on the locations and velocities of other cars (and other obstacles) on the road. For treating diseases, we want to find the proper dosage y of the corresponding medicine (and, it needed, the corresponding medicine) depending on the parameters $x_1, \ldots, x_n$ that characterize the patient: his/her age, weight, body temperature, blood pressure, etc.

Ideally, we should elicit, from the expert, the response y corresponding to all possible combinations of the values $x_1, \ldots, x_n$.

Need for approximate elicitation. Just like it is impossible to describe the values of the cumulative distribution function $F(x_1, \ldots, x_n)$ corresponding to all possible combinations of variables $x_1, \ldots, x_n$, it is also impossible to elicit the values $y(x_1, \ldots, x_n)$ corresponding to all possible combinations of inputs – for large n, there are billions of such possible combinations, and it is not realistically possible to ask billions of questions to the expert.

Thus, instead of eliciting, from the expert, the values y one-by-one, we need instead of elicit several rules that describe the expert's decisions.

Experts usually describe their rules by using imprecise ("fuzzy") words from natural language. Experts usually describe their knowledge in terms of if-then rules. For example, an expert describing his or her driving can formulate a rule like this: "if the car in front is close and starts somewhat slowing down, then the driver needs to gently hit the brakes".

Most such rules use imprecise ("fuzzy") words from natural language like "close", "somewhat", "gently", etc. It is therefore necessary to translate such rules into a precise computer-understandable control strategy. To perform such a translation, Lotfi A. Zadeh developed a special methodology that he called *fuzzy methodology*. Let us describe such methodology in detail.

The main stages of fuzzy methodology. The expert knowledge consists of several rules. Each rule is a propositional combination of several statements like "the car is close", "the car is somewhat slowing down", etc. Thus, to describe the expert knowledge, it is reasonable to first describe each such statement in precise terms, and then describe how to combine these statements.

First stage of fuzzy methodology. The first stage of fuzzy methodology is assigning meaning to individual statements such as "the car is close". The closeness depends in the distance d to the car.

For some values of this distance d, all experts agree that the car is close; for some other values of the distance, all experts would agree that the car is not close. However, for many other distances, experts may disagree – and moreover, they may not be 100% sure that the car is close or not close.

In the computer, true statements - i.e., statements about which we are absolutely sure – are represented by the boolean value "true", which in the computer is usually represented as 1. Similarly, absolutely false statements are represented, in the computer, by the boolean value "false", which is usually represented as 0. It is therefore reasonable to represent intermediate degree of certainty by the numbers intermediate between 0 and 1.

Thus, to describe what, e.g., "close" means, we need to describe, for each possible value d of the distance, the degree $\mu(d) \in [0,1]$ to which the distance d can be described as "close". The corresponding function is known as the *membership function*.

The value $\mu(d)$ can be obtained by simply asking an expert to mark his or her degree of certainty on a scale from 0 to 1. Alternatively, we can get a probability-type value by polling several expert and taking, as $\mu(d)$, the proportion of experts who believe that d corresponds to "close".

Second stage of fuzzy methodology: combining degrees of certainty. Most rules have several conditions, i.e., in other words, each such rule has a composite condition "$A_1(x_1)$ and $A_2(x_2)$ and ...", where A_i are the corresponding fuzzy properties like "close". To properly describe expert knowledge, it is therefore important to be able to describe the degree to which each such composite condition is satisfied.

Ideally, just like we we ask the expert, for each possible value of x_1, to what extend x_1 has the property A_1, we should also ask the expert, for each tuple $(x_1, x_2, \ldots)$, to what extend the given composite condition is satisfied for this tuple. However, as we have mentioned earlier, such elicitation is not realistically possible.

Since we cannot elicit this degree for all possible tuples, we need to estimate this degree based on what we know, i.e., based on the degrees $\mu_i(x_i)$ describing to what extent each value x_i satisfies the corresponding fuzzy property A_i.

In other words, we know the degree $a_1, \ldots, a_n$ to which different statements $A_i(x_i)$ are satisfied, and we need to come up with an estimate for a degree to which the "and"-combination $A_1(x_1) \,\&\, \ldots \,\&\, A_n(x_n)$ is satisfied. This estimate depends only on the values $x_1, \ldots, x_n$ and is, therefore, a function of these values. Let us denote this estimate by $f_\&(a_1, \ldots, a_n)$. This function is known as an *"and"-operation* or, alternatively, a *t-norm*.

What are the natural properties of an "and"-operation? First, for every two statements, $S \,\&\, S'$ usually means the same as $S' \,\&\, S$. Thus, it makes sense to require that we come up with the same estimates for the degrees to which these two combined statement hold, i.e., that for every two numbers s and s', we should have $f_\&(s, s') = f_\&(s', s)$. In mathematical terms, this means that the "and"-operation should be *commutative*.

Similarly, from the fact that $(S \,\&\, S') \,\&\, S''$ means the same as $S \,\&\, S' \,\&\, S''$ or $S \,\&\, (S' \,\&\, S'')$ means that the "and"-operation should be *associative*

$$f_\&(f_\&(s, s'), s'') = f_\&(s, f_\&(s', s''))$$

and, in general, that the combination of three or more degrees should be reduced to combining pairs:

$$f_\&(a_1, a_2, a_3, \ldots, a_n) = f_\&(\ldots (f_\&(f_\&(a_1, a_2), a_3), \ldots, a_n).$$

These are the usual requirements for a t-norm [15, 22].

The choice of an "and"-operation depends on the situation. In principle, there are many different "and"-operations. Out of all possible "and"-operations, we should select a one for which the resulting estimates are the closest to the experts' own estimates of the degrees of the composite statements $A_1(x_1) \,\&\, \ldots \,\&\, A_n(x_n)$. In other words, the "and"-operation should be determined empirically.

Historically first such determination was performed by the designers of the world's first practically successful expert system, a medical expert system MYCIN intended for diagnosing rare blood diseases; see, e.g., [6].

It is worth mentioning that the authors of MYCIN were initially under the impression that the "and"-operation that they discovered is universally applicable and thus, represents the general law of human reasoning.

They realized that the medicine-relevant "and"-operation is not universal when they started applying their general scheme to geophysics. It turned out that the medically best "and"-operation is not appropriate for geophysics at all. This makes sense:

- in search for oil – a typical application of geophysics – it makes sense to start drilling a well once there is a reasonable expectation that this well will be productive – and it is OK that a large portion of these wells do not produce, as long as on average, we are successful;
- in contrast, in medicine, we do not want to perform a serious surgery on a patient unless we are absolutely sure about the diagnosis.

In other words, in medicine, experts use very conservative estimates, while in geophysics, they use more optimistic ones.

So, different application domains use different "and"-operations – but the same "and"-operation is useful for all statements within a given application domain.

3 How the Main Ideas Behind Fuzzy Methodology Can Be Applied to Econometrics

Let us first try a straightforward application of fuzzy methodology. At first glance, in economics, we face exactly the same problem as on the second stage of the general fuzzy methodology. Namely, for each tuple $(x_1, \ldots, x_n)$:

- we know the degrees (probabilities) $F_i(x_i)$ with which the statements $X_i \le x_i$ hold, and
- we want to estimate the degree $F(x_1, \ldots, x_n)$ to which the combined statement $(X_1 \le x_1) \,\&\, \ldots \,\&\, (X_n \le x_n)$ holds.

If we apply the general fuzzy methodology to this problem, we end up with the following estimate for the joint distribution:

$$F(x_1, \ldots, x_n) \approx f_\&(F_1(x_1), \ldots, F_n(x_n))$$

for an appropriate "and"-operation $f_\&(a, b)$.

Alas, this straightforward application does not work. Unfortunately, in many economic problems, the above straightforward application of fuzzy logic does not work well.

Specifically, in principle, we can determine the corresponding "and"-operation by selecting a pair of inputs (i, j) and selecting a function $f_\&(a, b)$ which is the best fit for the corresponding 2-D case $F_{i,j}(x_i, x_j) \approx f_\&(F_i(x_i), F_j(x_j))$, where

$$F_{i,j}(x_i, x_j) \stackrel{\text{def}}{=} \text{Prob}(X_i \le x_i \,\&\, X_j \le x_j)$$

is the 2-D marginal distribution. What we find is that for different pairs (i, j), we thus get drastically different "and"-operations, and when we use one of these operations to combine all n inputs $F_i(x_i)$, the result is not a good approximation to $F(x_1, \ldots, x_n)$ at all.

Why we get different "and"-operations. The fact that we get different "and"-operations for different pairs of variable (i, j) is easy to explain. We have mentioned, in the previous section, that in different application domains, e.g., in medicine and geosciences (and oil exploration), expert decision making is described by different 'and"-operations. But to properly describe the economy, we need to take into account both medicine and oil exploration. So inevitably, we need to use different "and"-operations to describe different phenomena contributing to the economy as a whole.

Need to consider fuzzy logics with several different "and"-operations. Thus, we arrive at the need to consider fuzzy logics with several different "and"-operations.

This possibility is sometimes discussed on a theoretical level (see, e.g., [22]), when a formal description of a fuzzy logic contains two "and"-operations: $\min = \wedge$ and the actual t-norm &. However, here, we have a very practical need for such different operations.

For the case three variables, the use of different operations means that, in principle, we should use one particular combination operation $C_{12}(u, v)$ to combine the conditional probabilities $F_1(x_1 \,|\, x_3)$ and $F_2(x_2 \,|\, x_3)$ and other combination operations to combine, e.g., $F_1(x_1)$ and $F_3(x_3)$ into a joint distribution $F_{13}(x_1, x_3) = C_{13}(x_1, x_3)$ based on which we can compute the corresponding conditional probabilities $F_1(x_1 \,|\, x_3)$. This is exactly what we have in vine copulas.

And indeed, in several specific applications, we get the best fit with the empirical data if we consider different combination operations for different pairs of variables [11].

Need to go beyond associative "and"-operations: empirical evidence. In many practical problems, associative "and"-operations lead to a good description of the corresponding economic phenomenon. However, in some other cases, a non-associative operation leads to a better approximation [11].

This empirical conclusion is in line with the fact that our actual reasoning is often described by a non-associative "and"-operation; see, e.g., [28].

Need to go beyond associative "and"-operations: an explanation. When the same "and"-operation is used to combine all the statements, the equivalence of the statements $S \,\&\, (S' \,\&\, S'')$ and $(S \,\&\, S') \,\&\, S''$ naturally leads to the associativity requirement. However, once we assume that the "and"-operations for combining different pairs may be different, this equivalence no longer implies associativity of each of these "and"-operations.

This explanation is line with the fact that some naturally emerging "and"-operations are indeed non-associative [5, 10, 16, 26].

So, we need several possible non-associative "and"-operations. Thus, to adequately describe economic phenomena, we need to use several possible non-associative "and"-operation.

The resulting extension of fuzzy logic is, in effect, vine copulas. Thus, fuzzy ideas explain why vine copulas are successful in describing economic phenomena.

4 Conclusions

One of the most empirically successful way to describe economic phenomena is by using a special way of approximately describing dependence between multiple variables known as vine copulas. The problem with vine copulas is that they constitute a semi-heuristic technique, it is not clear why exactly this way of approximating multi-dimensional distributions works so well. In this paper, we have shown that vine copulas naturally appear if we apply an economics-justified generalization of the traditional fuzzy methodology: namely, a generalization in which there are several different "and"-operations, and each of these operations is not necessarily associative.

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