Standing on the Shoulders of the Giants: Why Constructive Mathematics, Probability Theory, Interval Mathematics, and Fuzzy Mathematics Are Important

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Abstract

Recent death of Ray Moore, one of the fathers of interval mathematics, inspired these thoughts on why interval computations – and several other related areas of study – are important, and what we can learn from the successes of these areas' founders and promoters.

The end of an era. On April 1, 2015, the interval computation community was saddened to learn that Ramon "Ray" Moore, one of the founding fathers of interval mathematics, is no longer with us. He has always been very active. And he was special. Many researchers come up with interesting and useful results, but not too many found a new direction of mathematics, direction with hundreds of followers. What made this particular direction different?

What are the main objectives of science and engineering? What was different about interval mathematics, why this particular idea became successful in many applications? To understand this success, let us recall what are the main objectives of science and engineering in general.

Of course, there is intellectual curiosity: we want to understand why the sky is blue, why the Sun shines, what causes rain and what causes earthquakes. This is what motivates Newtons and Einsteins. However, for the majority of people, the most important objective is to *predict* future events, and to come up with ways to make future events more beneficial to us humans. For most people, the main reason for studying what causes rain is to be able to predict when it will rain. The main reason for studying how viruses infect a person and how they interact with different cells and different chemicals is to be able to predict how the patient will feel if we try a certain medicine – and ideally, to come up with a medicine that will make the patient recover as soon as possible. The main reason for studying celestial mechanics is to predict where a planet

– or a spaceship – will be, and to use this knowledge to come up with the best spaceship trajectories.

How are these objectives attained now? Physicists uncover the *physical laws*, i.e., the relations between the past, current, and future values of different physical quantities. Once these laws are given – in terms of differential equations, in terms of operator equations, in whatever else form – we then try to use these laws to come up with algorithms that, given the current and past observations, enable us to predict the desired future values of the quantities of interest – and to find parameters of trajectories and constructions that optimize the future values of the corresponding objective functions.

When we design and apply these algorithms, it is important to take into account that we usually have only *partial* knowledge about the current and past states of the world. Indeed, this information comes either from measurements or from expert estimates; expert estimates are especially important in areas where direct measurements are difficult, e.g., in medicine and in geology (where it is difficult to perform measurements inside a human body or inside the Earth). Measurements are never absolutely accurate, and expert estimates are even less accurate.

First task and the resulting emergence of constructive mathematics. Based on this, what are our main tasks? Once the physicists have uncovered the physical laws, and mathematicians have proven that these laws are sufficient to predict the future values – i.e., that for each current state, there *exists* a unique future state satisfying these relations, we face the first important task: of coming up with the corresponding *algorithm*.

In other words, we need to move from a mathematical statement $\exists x\, P(x)$ to an algorithm that actually computes the corresponding object x. Of course, such algorithms have been developed in mathematics since the ancient times. Such algorithms are known for many problems. And eventually, a natural question emerged: instead of a case-by-case development of such algorithms, why not come with a general way of generating these algorithms?

Let us elaborate on this a little bit. From the practical viewpoint, existential statements for which no algorithms are possible are useless. Such pure-existence statements may be very interesting for pure mathematics, but for the corresponding practical problems, when we ask whether a given system of physical equations has a solution, we would like to come up with an algorithm for solving this system. From this viewpoint, it is desirable to come up with a version of mathematics in which $\exists x\, P(x)$ *means* that x can be algorithmically computed – and where from the proof of this statement, we can actually extract the appropriate algorithm. Such a version was indeed developed in the 1940s and 1950s, mostly by Andrei A. Markov (son of the author of Markov chains) and Nikolai A. Shanin, under the name *constructive* or *computable mathematics*; see, e.g., [1, 3, 4, 9, 22, 24, 40].

Second task: probability theory and interval mathematics. The next task is to take into account measurement uncertainty. In some cases, we know the probability of different values of measurement inaccuracy. Methods for dealing with such probabilistic uncertainty date back to Karl F. Gauss, who started this field by introducing the ideas of normal (Gaussian) distribution – one of the most frequent probability distributions – and of data processing under such uncertainty (Least Squares etc.).

At first, specific techniques were developed for specific cases, but very soon, the new mathematical theory emerged. Usually, the formulation of probability theory as a precisely defined area of mathematics is attributed to Andrei N. Kolmogorov and his famous 1933 book on mathematical foundations of probability theory [21].

However, in many other cases, we do not know the corresponding probabilities. In many such cases, all we know is the upper bound Δ on the absolute value of the measurement error. In this case, once we know the measurement result $\widetilde{x}$, the only information that we have about the actual (unknown) value x of the corresponding quantity is that this value belongs to the interval $[\widetilde{x}-\Delta, \widetilde{x}+\Delta]$. We therefore need to be able to take this interval uncertainty into account.

Again, people have been dealing with interval-type uncertainty for ages, it can traced to Archimedes providing bounds for π [2]. But eventually, an idea occurred: instead of doing it on a case-by-case basis, why not come up with a general way of taking this interval uncertainty into account? In other words, instead of first coming up with an algorithm for processing exact numbers and then thinking how to modify this algorithm so that it will take uncertainty into account, why not come up with methods that would enable us to directly design interval-processing algorithms? This was the main idea behind Moore's *interval mathematics* [26, 27, 28, 29, 30, 32, 34, 35] – which was independently developed by T. Sunaga and M. Warmus [37, 38, 39]; see also [25, 30] for the history of interval mathematics and [19, 33] for its current state.

Moore started with *interval arithmetic*, i.e., with showing how simple arithmetic operations will look like under interval uncertainty, in other words, what will be the intervals of possible values for $a+b$, $a-b$, $a\cdot b$, etc., when we know intervals of possible values of a and b. Later on, Moore and others developed more complex techniques, but the corresponding formulas of interval arithmetic remain the basis of most interval techniques.

Third task: fuzzy mathematics. The remaining task is to take into account uncertainty of expert estimates. Again, this is something that people have been doing for ages. But an idea naturally appeared: instead of trying to capture expert knowledge and expert uncertainty on a case-by-case basis, why not come up with a general way to describe such an uncertainty? This was the main idea behind Lotfi A. Zadeh's *fuzzy mathematics* [43]; see also [20, 36].

Specifically, Zadeh came up with an idea of how to describe expert uncertainty – which experts usually describe by using imprecise ("fuzzy") words from natural language (like "somewhat") – in terms that a computer can understand and process. His natural idea was that since in the computer, "true" is usually represented as 1, and "false" as 0, we can use intermediate numbers (i.e., numbers from the interval $[0,1]$) to describe different degrees of expert certainty.

Later, this idea was developed further, with the possibility of using more complex degrees of certainty, but the interval $[0, 1]$ remains the basic foundations of fuzzy techniques.

This is why. In our opinion, this is what explains the success of interval mathematics – as well as the success of constructive mathematics, probability theory, and fuzzy mathematics: that interval mathematics is aimed at solving one of the several fundamental problems of science and engineering applications.

But is this all new? As the Bible teaches us, there is nothing completely new under the Sun. Yes, we – following Newton's famous phrase – we stand on the shoulders of the giants, but these giants themselves were standing on the shoulders of others – in the sense that they used mathematical results developed before them.

From the purely mathematical viewpoint, the 1950s constructive mathematics is largely equivalent to the *intuitionistic mathematics* developed by Brouwer by the 1920s; see, e.g., [10, 11, 12, 14, 15, 16, 18]. Kolmogorov's mathematical foundations of probability theory simply describe probability as a measure μ for which the measure of the whole space is 1 – and measure theory was developed way before Kolmogorov, by Lebesgue and others. Formulas for interval arithmetic and even some rudimentary ideas of interval mathematics can be traced to several 1930s sources; see, e.g., [5, 6, 7, 8, 17, 41, 42]. And the idea of using the interval $[0, 1]$ to describe degrees of truth can be traced back to the 1920s papers by Lukasiewicz [23].

It is all new. Yes, in all these cases, the pure mathematical formalism is rather trivial and not new: measure theory was known way before Kolmogorov, intuitionistic mathematics was invented before Markov and Shanin, operations with intervals have been explicitly formulated in many previous papers, and min and max operations as "and" and "or" have been known since the 1920s. However, it is all new if we look beyond pure mathematics, to the corresponding application problems.

Yes, measure theory originated with Lebesque, but Kolmogorov was the first to show that many somewhat informal general results of probability theory can be derived from measure theory. Yes, intuitionistic mathematics was known since 1920s, but Markov and Shanin were the first to show that it can be used to analyze what can be algorithmically computed. Arithmetic operations with intervals were known for a long time, but Moore (as well as Sunaga and Warmus) was among the first to provide general algorithms using interval arithmetic to estimate the range of a generic functions – from the simplest idea of "naive" (straightforward) interval arithmetic, when we simply replace each elementary arithmetic operation with the corresponding operation with intervals, to more efficient schemes like the centered form; see, e.g., [19, 33]. Yes, the logic on the interval $[0, 1]$ has been known for decades, but Zadeh was the first one who used it to design a general methodology for translating expert knowledge formulated by using imprecise ("fuzzy") words from natural language into precise computer-understandable terms.

Let me give you one more example: the General Relativity theory is credited to Einstein – in my opinion, absolutely correctly. Not many people outside physics know that the famous mathematician David Hilbert (of Hilbert's problems fame) independently came up with the same equations as Einstein – his paper was submitted two weeks after Einstein's and published two weeks after Einstein's. If Hilbert's paper was submitted two weeks earlier, would he then get all the fame? From the purely mathematical viewpoint, yes: he would then be the first to come up with equations. However, from the physical viewpoint, he would only get *some* credit. If you look at his paper, the equations is all he did, while Einstein analyzed physical consequences of these equations – something that enabled the experiments to check his theory.

In addition to research, leadership is also very important. Yes, research results are important, but it is also important to promote these results. Researchers often think that once a good idea is published, people will jump on it and start using it. Sometimes, they do, but in many cases, a relentless promotion and explanation of a new idea is needed for this idea to catch up. Many researchers do not do it: it takes time away from research, and it sounds immodest if you promote your own idea too much. But without such a promotion, ideas often just die – or, to be more precise, wait until someone else, with a better promotion skills, rediscovers these ideas.

And this is where true leadership shows. Markov and Shanin spent a lot of time promoting the construcivism ideas, cultivating students, answering criticisms, patiently trying to reformulate their ideas in a more and more clear form. In their days, hardly anyone outside logics knew about intuitionistic logic, but it was difficult to find a mathematician in St. Petersburg or Moscow who has never heard about constructive mathematics. They may have disagreed with it, they may have misconception about it, but they knew about it.

Similarly, not many people heard about Bradis or even about Sunaga or Warmus – but many researchers and practitioners heard about interval mathematics. They may disagree with it, they may have misconceptions about it ("I tried interval methods, they do not work"), but most have heard about it, and they have heard about Moore. Why? Because Moore was the one relentlessly promoting its ideas, publishing books and papers, attending conferences, fighting the criticisms. He was very active on the interval mailing list. Sometimes, he expressed his ideas and opinions openly. But often, he felt that it is more appropriate for someone who is more knowledgeable in a certain application area to reply, sometimes quietly, to clarify misunderstandings. Even a few weeks before his untimely death, he asked me – since I also know fuzzy techniques – to look into a fuzzy-related paper that showed a misunderstanding of interval methods (yes, along the usual lines "I tried interval methods, they do not work", which usually means that naive interval methods lead to a huge overestimation).

Not many people outside logic know about Lukasiewicz, but everyone knows about fuzzy – and about Zadeh, because Lotfi Zadeh used to tirelessly promote his ideas – and ideas of others who enhanced and applied his techniques.

This is their under-appreciated contribution, without which success of others, success of applications would not be possible – we may have laughed at Shanin standing up at every seminar to ask what is computable and what is not, we may have laughed at Zadeh for repeating the same ideas again and again – but who is laughing now: this repetition worked!

Terminology is important. In all these cases, one of the important elements of success was the right term.

The term "intuitionism" does not apply to a mathematician, it smacks of intuition, something imprecise, something non-mathematical. In contract, "constructivism" mean constructed, a very mathematical term – after all, geometric constructions is one of the main origins of mathematics – which also conveys the idea of computability.

Similarly, "interval mathematics", "interval computations" are clear and catchy terms, immediately conveying the meaning of the field.

And "fuzzy", the term selected to bring on a controversy – since "fuzzy thinking" is an English term for bad thinking – spread because of its catchiness.

There are terms that attract – and thus make whatever called by this term more attractive. Socialism – something supposedly beneficial to the society – sounds good, and this sounding part partly explains its appeal, as opposed to capitalism, which does not sound as good. Impressionism – a large part of its appear comes from its name. Coming up with such names is not easy, and this is part of the genius of the giants who started these fields.

So where do we go from here: we need to learn from the giants. We cannot all be giants, but we can learn from them. In my opinion, the main lesson is that we need to relentlessly promote important ideas – we need to learn to do it better, we need to learn not to hesitate to do it, and we need to appreciate it when others are doing it. Only then will the ideas propagate – as they should, only then the progress will come.

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