Standing on the Shoulders of the Giants: Why Constructive Mathematics, Probability Theory, Interval Mathematics, and Fuzzy Mathematics Are Important*

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Abstract

The recent death of Ray Moore, one of the fathers of interval mathematics, inspired these thoughts on why interval computations — and several related areas of study — are important, and what we can learn from the successes of these areas' founders and promoters.

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The end of an era. On April 1, 2015, the interval computation community was saddened to learn that Ramon "Ray" Moore, one of the founding fathers of interval mathematics, is no longer with us. He was always very active. And he was special. Many researchers obtain interesting and useful results, but few originate new directions in mathematics that attract hundreds of followers. What made this particular direction different?

What are the main objectives of science and engineering? What was different about interval mathematics? Why did this particular idea succeed in so many applications? To understand this success, let us consider the main objectives of science and engineering in general.

Of course, there is intellectual curiosity: we want to understand why the sky is blue, why the Sun shines, and what causes earthquakes and rain. This is what motivates Newtons and Einsteins. For the majority of people, however, the most important objective is to *predict* and to favorably *influence* future events. For most people, the main reason for studying the causes of rain is to be able to predict rain. The main reason for studying how viruses infect the body and how they interact with different cells and chemicals is to be able to predict how patients will feel if we try a certain

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medicine — and ideally, to discover a medicine that will help speed recovery. The main reason for studying celestial mechanics is to predict where a planet — or a spaceship — will be, and to use this knowledge to formulate an optimal trajectory.

How are these objectives attained now? Physicists uncover the *physical laws*, i.e., the relations between the past, present, and future values of certain physical quantities. Once these laws are given — in terms of differential equations, operator equations, or other mathematical entities — we try to use them to develop algorithms that, given the present and past observations, enable us to predict the desired future values of the quantities of interest and to find parameters of trajectories and constructions that optimize the future values of the corresponding objective functions.

In designing and applying these algorithms, it is important to take into account that we usually have only *partial* knowledge about the present and past states of the world. Indeed, this information comes either from measurements or from expert estimates; the latter are especially important in areas where direct measurements are difficult, e.g., in medicine and in geology (where it is difficult to perform measurements inside a human body or inside the Earth). Measurements are never absolutely accurate, and expert estimates are even less accurate.

First task and the resulting emergence of constructive mathematics. Based on this, what are our main tasks? Once the physicists have uncovered the physical laws, and mathematicians have proven that these laws are sufficient to predict the future values, i.e., that for each present state there *exists* a unique future state satisfying these relations, we face the first important task: producing an appropriate *algorithm*.

In other words, we need to move from a mathematical statement $\exists x\, P(x)$ to an algorithm that actually computes the corresponding object x. Of course, such algorithms have been developed in mathematics since ancient times and are available for many problems. A natural question eventually emerged: instead of a case-by-case development of such algorithms, why not seek a general way of developing them?

Let us elaborate on this a little bit. From a practical standpoint, existential statements for which no algorithms are possible are useless. Such pure-existence statements may be fascinating to pure mathematics, but for the corresponding practical problems, when we ask whether a given system of physical equations is solvable, we would prefer to have an algorithm for obtaining a solution. In this sense it is desirable to have a version of mathematics in which $\exists x\, P(x)$ *means* that x can be algorithmically computed, and where the proof of this statement actually yields an appropriate algorithm. Such a version was indeed developed in the 1940s and 1950s, mostly by Andrei A. Markov (son of the author of Markov chains) and Nikolai A. Shanin, under the name *constructive* or *computable mathematics*; see, e.g., [1, 3, 4, 9, 22, 24, 40].

Second task: probability theory and interval mathematics. Next we should take into account measurement uncertainty. In some cases we know the probability of different values of measurement inaccuracy. Methods for dealing with such probabilistic uncertainty date back to Karl F. Gauss, who spawned this field by introducing the ideas of the normal (Gaussian) distribution — one of the most common probability distributions — and of data processing under such uncertainty (least squares, etc.).

At first, specific techniques were developed for specific cases, but very soon a new mathematical theory emerged. The formulation of probability theory as a precisely defined area of mathematics is commonly attributed to Andrei N. Kolmogorov and his famous 1933 book on mathematical foundations of probability theory [21].

However, we often do not know the corresponding probabilities. We may simply have an upper bound Δ on the absolute value of the measurement error. In this case, once we know the measurement result $\widetilde{x}$, the only information we have about the actual (unknown) value x of the corresponding quantity is that it lies in the interval $[\widetilde{x}-\Delta, \widetilde{x}+\Delta]$. Hence we must be able to take this interval uncertainty into account.

Again, people have been dealing with interval-type uncertainty for ages; it can traced to Archimedes providing bounds for π [2]. But eventually an idea occurred: instead of doing it on a case-by-case basis, why not invent a general way of accounting for interval uncertainty? In other words, instead of first producing an algorithm for processing exact numbers and then trying to modify it to account for uncertainty, why not seek methods that would enable us to directly design interval-processing algorithms? This was the main idea behind Moore's *interval mathematics* [26, 27, 28, 29, 30, 32, 34, 35], which was independently developed by T. Sunaga and M. Warmus [37, 38, 39]; see also [25, 30] for the history of interval mathematics, and [19, 33] for its current status.

Moore started with *interval arithmetic*, i.e., with showing how simple arithmetic operations appear under interval uncertainty. In other words, what will be the intervals of possible values for $a+b$, $a-b$, $a\cdot b$, etc., when we know intervals of possible values for a and b? Moore and others later developed more complex techniques, but the corresponding formulas of interval arithmetic remain the basis of most interval techniques.

Third task: fuzzy mathematics. The remaining task is to take into account uncertainties in expert estimates. Again, this is something that people have been doing for ages. But an idea naturally appeared: instead of trying to capture expert knowledge and expert uncertainty on a case-by-case basis, why not seek a general way of describing such uncertainty? This was the main idea behind Lotfi A. Zadeh's *fuzzy mathematics* [43]; see also [20, 36].

Specifically, Zadeh showed how to describe expert uncertainty — which experts usually describe by using imprecise ("fuzzy") words from natural language (like "somewhat") — in terms that computers can understand and process. His natural idea was that since computers usually represent "true" by 1 and "false" by 0, we can use intermediate numbers (i.e., numbers from the interval $[0, 1]$) to describe different degrees of expert certainty.

Later this idea was developed further, with the possibility of using more complex degrees of certainty, but the interval $[0, 1]$ remains at the basic foundation of fuzzy techniques.

This is why. In our opinion, this is what explains the success of interval mathematics, as well as the success of constructive mathematics, probability theory, and fuzzy mathematics: that interval mathematics is aimed at solving one of several fundamental problems in science and engineering application.

But is this all new? The Bible states that there is nothing completely new under the Sun. Yes, we — following Newton's famous phrase — stand on the shoulders of giants, but these giants themselves were standing on the shoulders of others, in the sense that they used mathematical results developed before them.

From a purely mathematical viewpoint, the 1950s constructive mathematics is largely equivalent to the *intuitionistic mathematics* developed by Brouwer by the 1920s; see, e.g., [10, 11, 12, 14, 15, 16, 18]. Kolmogorov's mathematical foundations of probability theory simply describe probability as a measure μ for which the measure of the whole space is 1 — and measure theory was developed well before Kolmogorov, by Lebesgue and others. Formulas for interval arithmetic and even some rudimentary ideas of interval mathematics can be traced to several 1930s sources; see, e.g., [5, 6, 7, 8, 17, 41, 42]. And the idea of using the interval $[0, 1]$ to describe degrees of truth can be traced to the 1920s papers by Lukasiewicz [23].

It is all new. Yes, in all these cases the pure mathematical formalism is rather trivial and not new: measure theory was known long before Kolmogorov, intuitionistic mathematics was invented before Markov and Shanin, operations with intervals were explicitly formulated in many previous papers, and min and max operations as "and" and "or" were known since the 1920s. However, it is all new if we look beyond pure mathematics, to the corresponding application problems.

Yes, measure theory originated with Lebesgue, but Kolmogorov was the first to show that many somewhat informal general results of probability theory can be derived from measure theory. Yes, intuitionistic mathematics was known since 1920s, but Markov and Shanin were the first to show that it can be used to analyze what can be algorithmically computed. Arithmetic operations with intervals were known for a long time, but Moore (as well as Sunaga and Warmus) was among the first to provide general algorithms using interval arithmetic to estimate the range of generic function — from the simplest idea of "naive" (straightforward) interval arithmetic, when we simply replace each elementary arithmetic operation with the corresponding operation with intervals, to more efficient schemes like the centered form; see, e.g., [19, 33]. Yes, logic on the interval $[0, 1]$ has been known for decades, but Zadeh was the first to use it to design a general methodology for translating expert knowledge formulated using imprecise ("fuzzy") words from natural language into precise computer-understandable terms.

Let me offer one more example: General Relativity theory is credited to Einstein — in my opinion, absolutely correctly. Not many people outside physics know that the mathematician David Hilbert (of Hilbert's problems fame) independently obtained the same equations as Einstein; his paper was submitted two weeks after Einstein's and published two weeks after Einstein's. If Hilbert's paper had been submitted two weeks earlier, would he have been given all the credit? From a purely mathematical viewpoint, yes: he would have been the first to come up with the equations. However, from the physical viewpoint he would only get *some* of the credit. Examination of Hilbert's paper reveals work limited to the equations themselves, whereas Einstein analyzed their physical consequences — something that enabled the experiments to check his theory.

Research is important, but so is leadership. Yes, research results are important, but so is their effective promotion. Researchers often expect that once a good idea is published, people will immediately start using it. Sometimes they do, but in many cases relentless promotion and explanation of a new idea are required for widespread adoption. Many researchers shy away from such promotion: it takes time away from research, and it sounds immodest if you promote your own idea too much. But without such promotion ideas often simply die or lie dormant until someone else, with better promotion skills, rediscovers them.

And this is where true leadership is shown. Markov and Shanin spent much time promoting the constructivism ideas: cultivating students, answering criticisms, patiently trying to reformulate their ideas in increasingly clear forms. In their day, hardly anyone outside logic knew about intuitionistic logic, but it was difficult to find a mathematician in St. Petersburg or Moscow who had never heard about constructive mathematics. They may have disagreed with it, they may have had misconceptions about it, but they knew about it.

Similarly, not many people have heard about Bradis — or even about Sunaga or Warmus — but many researchers and practitioners have heard about interval mathematics. They may disagree with it, or have misconceptions about it ("I tried interval methods and they don't work"), but most have heard of it and they have heard of Moore. Why? Because Moore was the one relentlessly promoting his ideas: publishing books and papers, attending conferences, fighting the criticisms. He was very active on the interval mailing list. Sometimes he expressed his ideas and opinions openly. But often he felt it more appropriate for someone more knowledgeable in a certain application area to reply, sometimes quietly, to clarify misunderstandings. Even a few weeks before his untimely death, he asked me — since I also know fuzzy techniques — to look into a fuzzy-related paper that showed a misunderstanding of interval methods (yes, along the usual lines of "I tried interval methods and they do not work", which usually means that naive interval methods lead to unacceptable overestimation).

Not many people outside logic know about Lukasiewicz, but everyone knows about fuzzy — and about Zadeh, because Lotfi Zadeh used to tirelessly promote his ideas — and the ideas of others who enhanced and applied his techniques.

This is their underappreciated contribution, without which the successes of others — and particularly successes in applications — would not be possible. We may have laughed at Shanin standing up at every seminar to ask what is computable and what is not; we may have laughed at Zadeh for repeating the same ideas again and again. But who is laughing now: this repetition worked!

Terminology is important. In all these cases, one of the key elements of success was the right choice of words.

The term "intuitionism" does not befit a mathematician; it smacks of intuition, something imprecise, something unmathematical. In contract, "constructivism" is quite mathematical sounding (indeed geometrical construction is one of the main origins of mathematics) and also conveys the idea of computability.

Similarly, the terms "interval mathematics" and "interval computations" are clear and catchy, immediately conveying the meaning of the field.

And "fuzzy", the term selected to bring on controversy (since "fuzzy thinking" is an English term for bad thinking) spread because of its catchiness.

Certain terminology is attractive and lends attractiveness to whatever it denotes. Socialism — something supposedly beneficial to the society — may sound good and this may partly explain its appeal. Capitalism, in contrast, may not sound as good. Impressionism is a powerful name for an approach. Coming up with such names is not easy, and this is part of the genius of the giants who started these fields.

So where do we go from here: we need to learn from the giants. We cannot all be giants, but we can learn from them. In my opinion, the main lesson is that we must relentlessly promote important ideas. We must learn to do it better, without hesitancy, and to appreciate when others are doing it. Only then will the ideas propagate as they should. Only then will progress come.

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