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HOW TO DIVIDE A TERRITORY: AN ARGUMENT IN FAVOR OF PRIVATE PROPERTY

Dividing a disputed territory: a real-life problem. In many real-life situations, from conflicts between neighbors to conflicts between states, there is a dispute over a territory, as a result of which none of the sides can use this territory efficiently. In such situations, it is desirable to come up withe mutually beneficial agreement.

Our approach: Nash's bargaining solution. Conflict situations in which there are solutions which are better that status quo for all the participants are known as *cooperative games*. Such games were analyzed almost immediately after the emergence, in the 1940s, of *game theory* – methodologies for solving conflict situations.

In 1951, a future Nobelist John Nash showed that under certain reasonable assumption, the optimal solution if the one for which the product of the utilities is the largest possible. This solution is known as *Nash's bargaining solution* [2–4].

How to divide a territory: what is known. In [1, 5], Nash's bargaining solution is applied to the problem of dividing a disputed territory. Let X be the territory. For each point $x \in X$, let $u_i(x)$ denote the utility of the location x to the i-th participant. We need to divide the set X into n disjoint subsets S_i: $\bigcup\limits_{i=1}^{n} S_i = X$ and $S_i \cap S_j = \emptyset$ for $i \neq j$.

If we allocate, to the i-th participant, a part $S_i \subseteq X$ of this disputed territory, the utility u_i of this participant will be equal to $u_i = \int_{S_i} u_i(x)\,dx$. Nash's bargaining solution means that we select a division of the original territory X into sets $S_1, \ldots, S_n$ for which the product $\prod\limits_{i=1}^{n} u_i$ is the largest possible.

The solution to this optimization problem is as follows: we select some threshold values t_i, and assign a point $x \in X$ to the set S_i for which the ratio $u_i(x)/t_i$ is the largest possible. We select the values t_i for which the product $\prod\limits_{i=1}^{n} u_i$ is the largest possible.

In particular, for the case $n = 2$, when a conflict has only two sides, we have a threshold $t \stackrel{\text{def}}{=} t_1/t_2$ for which $x \in S_1$ if and only if $u_1(x)/u_2(x) \geq t$.

Why not joint control? The above formalization assumes that every location in the disputed territory should be allocated to one of the sides. But if several sides have interest, why not propose a joint control?

This has been done on the past in many areas: there was a joint British-Egyptian control of Sudan, joint allied control over Austria, over parts of Germany, etc.

Formalization of the new optimization problem. In this new formulation, instead of allocating a location x to one of the n sides of the conflict, we need to come up with the weights $w_i(x)$ that describe the degree of control of the i-th side over this location. These weights should, of course, add up to 1: $\sum\limits_{i=1}^{n} w_i(x) = 1$.

In this new formulation, the utility u_i of the i-th participant is equal to $u_i = \int_X w_i(x) \cdot u_i(x)\,dx$, and we need to select functions $w_i(x)$ for which

$$u = \prod_{i=1}^{n} u_i = \prod_{i=1}^{n} \left(\int_X w_i(x) \cdot u_i(x)\,dx \right)$$

is maximized under the constraint $\sum\limits_{i=1}^{n} w_i(x) = 1$ for each $x \in X$.

Solving the new optimization problem. Lagrange multiplier method reduces the above constraint optimization problem to the following unconstrained problem

$$J \stackrel{\text{def}}{=} \prod_{i=1}^{n} \left(\int_X w_i(x) \cdot u_i(x)\,dx \right) + \int \lambda(x) \cdot \left(\sum_{i=1}^{n} w_i(x) - 1 \right) dx \to \max_{w_i(x)}$$

for $w_i(x) \in [0, 1]$. When the optimal value $w_i(x)$ is inside the range $[0, 1]$, i.e., when $0 < w_i(x) < 1$, the derivative of the objective function J with respect to $w_i(x)$ must be equal to 0. So, for such locations x, we get $C_i \cdot u_i(x) + \lambda(x) = 0$, where we denoted $C_i \stackrel{\text{def}}{=} \prod\limits_{j \neq i} u_j$.

From $0 < w_i(x) < 1$ and $\sum\limits_{j=1}^{n} w_j(x) = 1$, it follows that there is at least one other participant k for which $0 < w_k(x)$ – and this, $0 < w_k(x) < 1$. For this k, we similarly have $C_k \cdot u_k(x) + \lambda(x) = 0$, hence $u_i(x)/u_k(x) = C_k/C_i$.

Conclusion. So, all the points for which $0 < w_i(x) < 1$, i.e., all the points of joint control, must be located on one the areas

$$\{x : u_i(x)/u_k(x) = C_k/C_i\}.$$

For generic functions $u_i(x)$ and for each constant C_k/C_i, this area has co-dimension 1 – a 0-dimensional point in a 1-D line, a 1-D line in a 2-D space, etc. Thus, this area has measure 0. So, locations from this area do not contribute to the corresponding integrals u_i and can, thus, be safely ignored.

And if we have a whole block of locations x for which $u_i(x)/u_k(x) =$ const, then instead of partial control we can as well divide this region between the i-th and the k-th participants, the utilities will not change.

For all other locations, we have $w_i(x) = 0$ or $w_i(x) = 1$. Thus, *joint control is never optimal*: Nash's bargaining solution implies that each location x is assigned to one of the participants.

Comment. In the above text, we talked about dividing a disputed territory, but the same argument can be repeated in other practical situations when we have an ownership dispute. For such general situations, our conclusion provides an argument for private property (as opposed to communal one).

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