Why Min-Based Conditioning

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Abstract. In many practical situations, we do not have full information about which alternatives are possible and which are not. In such situations, an expert can estimate, for each alternative, the degree to which this alternative is possible. Sometimes, experts can produce numerical estimates of their degrees, but often, they can only provide us with qualitative estimates: they inform us which degrees are higher, but do not provide us with numerical values for these degrees. After we get these degrees from the experts, we often gain additional information, because of which some alternatives which were previously considered possible are now excluded. To take this new information into account, we need to appropriately update the corresponding possibility degrees. In this paper, we prove that under several natural requirements on such an update procedure, there is only one procedure that satisfies all these requirements – namely, the min-based conditioning.

Keywords: imprecise knowledge, possibility distribution, conditioning, knowledge update, invariance

1. Formulation of the Problem

Need for ordinal-scale possibility degrees. It is often useful to describe, for each theoretically possible alternative ω from the set of all *theoretically* possible alternatives Ω, to what extent this alternative is, in the expert's opinion, *actually* possible.

Often, the only information that we can extract from experts is the qualitative one: which alternatives have a higher degree of possibility and which have lower degree. In some cases, we have a linear order between possible degrees, so all we know is the order of different alternatives, from the least possible to the most possible.

In principle, we could just use this order to process this information, but computers have been designed to process numbers – and they are still much better in processing numbers. So, to speed up processing of this data, degrees of possibility are usually described by numbers $\pi(\omega)$ from the interval $[0, 1]$: the higher the degree of possibility of an alternative ω, the larger the value $\pi(\omega)$.

These numbers by themselves do not have an exact meaning, the only meaning is in the order. So, in principle, the same meaning can be described if we apply any strictly increasing transformation to the interval $[0, 1]$.

Usually, some of this freedom is eliminating by the convention that the largest degree of possibility is set to 1; we can always achieve this with an appropriate transformation. Such possibility degrees are known as *normalized*. Thus, we arrive at the following definition (see, e.g., (Dubois, Lang, and Prade, 1994; Dubois, Moral, and Prade, 1998; Dubois and Prade, 1998; Gutierrez et al., 2014)):

Definition 1. *Let Ω be a finite Universe of discourse. A* possibility distribution *is a function $\pi : \Omega \to [0,1]$ for which*

$$\max_{\omega \in \Omega} \pi(\omega) = 1.$$

Need for conditioning and normalization. Often, after we have learned the possibility degrees $\pi(\omega)$, we acquire an additional information, that only some of the original alternatives are actually possible. Let us denote the set of actually possible alternatives by $\Psi \subset \Omega$. How will this information change the possibility degrees? What the new values $\pi'(\omega)$?

Of course, now that we learn that only alternatives from the set Ψ are actually possible, we should set $\pi'(\omega) = 0$ for all $\omega \notin \Psi$. For all other alternatives $\sigma \in \Psi$, at first glance, it may sound reasonable to just retain the original possibility degrees, i.e., to take $\pi'(\omega) = \pi(\omega)$. However, we have an additional requirement, that the largest possibility degree should always be 1, and the above procedure this does not always guarantee this requirements.

For example, if:

- we started with $\pi(a) = \mu(b) = 0.5$ and $\pi(c) = 1.0$, and
- we learned that $\omega \in \Psi = \{a, b\}$,

then:

- if we simply take $\pi'(a) = \pi'(b) = 0.5$ and $\pi'(c) = 0$,
- the largest of the resulting three degrees is not equal to 1.

It is therefore necessary to *normalize* the resulting degrees $\pi'(\omega)$, i.e., to transform them into new degrees for which the largest is 1.

Definition 2. *By a* conditioning operator*, we mean a mapping $(\pi \mid \Psi)$ that:*

- *inputs a possibility distribution π on a set Ω and a non-empty set $\Psi \subseteq \Omega$, and*
- *returns a new possibility distribution for which $(\pi \mid \Psi)(\omega) = 0$ for all $\omega \notin \Psi$.*

What are the reasonable conditioning operators?

2. Analysis of the Problem

Let us describe the desired properties of the conditioning operator.

First property: impossible alternatives should not matter. A first reasonable requirement is that since alternatives $\omega \notin \Psi$ are excluded, their original possibility degrees should not affect the resulting degrees. In other words, if two original possibility distributions π and π' differ only by their values outside Ψ, then the conditioning should be the same.

C1. If $\pi_{|\Psi} = \pi'_{|\Psi}$, i.e., if $\pi(\omega) = \pi'(\omega)$ for all $\omega \in \Psi$, then

$$(\pi \mid \Psi) = (\pi' \mid \Psi).$$

Second property: order between possibility degree of different alternatives should not change. Another reasonable condition is that while the numerical values of possibility degrees may change, the order between these degrees should not change:

C2. If $\pi(\omega) < \pi(\omega')$ for some $\omega, \omega' \in \Psi$, then

$$(\pi \mid \Psi)(\omega) < (\pi \mid \Psi)(\omega').$$

C3. If $\pi(\omega) = \pi(\omega')$ for some $\omega, \omega' \in \Psi$, then

$$(\pi \mid \Psi)(\omega) = (\pi \mid \Psi)(\omega').$$

Third property: order between different possibility distributions should not change. One more condition is that if in one situation, we had consistently higher possibility degrees than in another situation, different situations, then this relation should be preserved after conditioning:

C4. If $\pi(\omega) \le \pi'(\omega)$ for all $\omega \in \Psi$, then

$$(\pi \mid \Psi)(\omega) \le (\pi' \mid \Psi)(\omega) \text{ for all } \omega \in \Psi.$$

Fourth property: an impossible alternative should remain impossible. Another condition is that if we add a new alternative with 0 degree of possibility (or, equivalently, delete an alternative with 0 possibility), it should not change anything, i.e., this alternative should still have 0 possibility after conditioning, and all other values after conditioning will not change:

C5. If $\pi(\omega_0) = 0$ for some $\omega_0 \in \Psi$, then

$$(\pi \mid \Psi)(\omega_0) = 0 \text{ and } (\pi_{|\Psi - \{\omega_0\}} \mid \Psi) = (\pi \mid \Psi)_{|\Psi - \{\omega_0\}}.$$

Final property: invariance. Finally, since the degrees are defined modulo an arbitrary 1-1 increasing function $T : [0,1] \to [0,1]$, the conditioning operator should also not change if we apply such a transformation. To describe this property, for each possibility distribution π, by $T\pi$, we denote a possibility distribution that results from applying T: $(T\pi)(\omega) \stackrel{\text{def}}{=} T(\pi(\omega))$. Then, the corresponding property takes the following form:

C6. For every monotonic 1-1-increasing function $T : [0,1] \to [0,1]$, we have

$$(T\pi \mid \Psi) = T(\pi \mid \Psi).$$

Now, we are ready to formulate our main result.

3. Main Result

Proposition. *The only conditioning operator that satisfies the properties* **C1**–**C6** *is the min-based operator (*Dubois and Prade, 1984*;* Hisdal, 1978*) for which:*

- $(\pi \mid \Psi)(\omega) = 1$ *when* $\omega \in \Omega$ *and* $\pi(\omega) = \max\limits_{\omega' \in \Omega} \pi(\omega')$*;*
- $(\pi \mid \Psi)(\omega) = \pi(\omega)$ *when* $\omega \in \Omega$ *and* $\pi(\omega) < \max\limits_{\omega' \in \Omega} \pi(\omega')$*; and*
- $(\pi \mid \Psi)(\omega) = 0$ *when* $\omega \notin \Psi$*.*

Discussion. The usual derivation of the min-based conditioning (see, e.g., (Dubois, Lang, and Prade, 1994)) is to interpret the degree $(A \mid B)$ as the maximal value for which $A \,\&\, B$ (with min as "and"-operation) has the same truth value as $(A \mid B) \,\&\, B$.

Our result shows that *maximality* can be replaces with *invariance* – which reflects the ordinal-scale character of the corresponding possibility degrees.

Proof.

1°. It is easy to show that the min-based operator satisfies the properties **C1–C6**.

To complete the proof, we need to prove that, vice versa,

- every conditioning operator that satisfies these five properties
- is indeed the min-based operator.

To prove this statement, we will consider two possible cases:

- the case when the set Ψ contains some alternative ω for which $\pi(\omega) = 1$, and
- the case when the set Ψ does not contain any alternative ω for which $\pi(\omega) = 1$.

2°. Let us first consider the case when the set Ψ contains some alternative ω for which $\pi(\omega) = 1$. In this case, the min-based formula leads to $(\pi \mid \Psi)(\omega) = \pi(\omega)$ for all $\omega \in \Psi$.

Let us show that this equality holds for all conditioning operators that satisfy the properties **C1**–**C6**.

2.1°. If there is no $\omega_0 \in \Psi$ for which $\pi(\omega_0) = 0$, let us add such an element to our set Ω. According to Property **C5**, this will not change the result. Thus, without losing generality, we can safely assume that there is an element $\omega_0 \in \Psi$ for which $\pi(\omega_0) = 0$.

As for the values $\pi(\omega)$ for $\omega \notin \Psi$, we can use the property **C1** to replace them with zeros.

2.2°. Let us sort values $\psi(\omega)$ corresponding to different alternatives $\omega \in \Psi$ in increasing order. We know that the resulting list of values includes 0 and 1, so this list has the form

$$v_1 = 0 < v_2 < \ldots < v_{k-1} < v_k = 1,$$

where k is the number of different values $\pi(\omega)$ corresponding to $\omega \in \Psi$.

Let us use property **C6** to prove that the values $(\pi \mid \Psi)$ should also be from this list. Indeed, let us consider the following strictly increasing function $T(v)$: for $v_i \le v \le v_{i+1}$, we take

$$T(v) = v_i + \left(\frac{v - v_i}{v_{i+1} - v_i}\right)^2 \cdot (v_{i+1} - v_i).$$

One can easily check that for this function, $T(v_i) = v_i$ for all i, so $T(\pi) = \pi$. Thus, the property **C6** implies that $T(\pi \mid \Psi) = (\pi \mid \Psi)$, i.e., that for each value $v = (\pi \mid \Psi)(\omega)$, we should have $T(v) = v$. But for the above function $T(v)$, the only such values are $v_1, \ldots, v_k$.

So, indeed, the values $v_1 < \ldots < v_k$ are mapped to the same k values. By properties **C2** and **C3**, equal values of $\pi(\omega)$ are mapped into equal values of $(\pi \mid \Psi)(\omega)$, and smaller values of $\pi(\omega)$ are mapped into smaller values of $(\pi \mid \Psi)(\omega)$. Thus, the values v'_i corresponding to v_i are also sorted in increasing order: $v'_1 < \ldots < v'_k$. Each new value v'_i must coincide with one of the original values v_j. So, in the increasing list $v_1 < \ldots < v_k$ of k values, we have k new values v'_i which have the same order. This implies that v'_1 must be the smallest of v_i, i.e., $v'_1 = v_1$, that v'_2 be the second smallest, i.e., $v'_2 = v_2$, and, in general, $v'_i = v_i$, i.e., indeed, $(\pi \mid \Psi)(\omega) = \pi(\omega)$ for all $\omega \in \Psi$.

3°. Let us now consider the case when the set Ψ does not contain some alternative ω for which $\pi(\omega) = 1$.

In this case, we can also add (if needed) an element ω_0 for which $\pi(\omega_0) = 0$, and sort the values $\pi(\omega)$ corresponding to $\omega \in \Psi$ into an increasing sequence $v_1 = 0 < v_2 < \ldots < v_k < 1$; the only difference is that in this case, the largest value v_k in this increasing sequence is smaller than 1.

Similarly to Part 2 of this proof, we can prove that each of the the values v_i maps into one of the values $v_1, \ldots, v_k$, or 1, and that if $v_i < v_j$, then $v'_i < v'_j$.

Let us consider a new possibility measure π' that is equal to 1 when $\pi(\omega) = v_k$ and which coincides with π for all other ω. From Part 2 of this proof, we know that $(\pi' \mid \Psi)(\omega) = \pi'(\omega)$. Here, $\pi(\omega) \le \pi'(\omega)$ for all $\omega \in \Psi$, so by property **C4**, we have $(\pi \mid \Psi)(\omega) \le (\pi' \mid \Psi)(\omega) = \pi'(\omega)$ for all $\omega \in \Psi$. So, in our notations, we have $v'_i \le v_i$ for all $i \le k - 1$.

For $i = 1$, we have $v'_1 \le v_1 = 0$, so $v'_1 = 0$. For $i = 2$, we have $v'_2 \le v_2$. Since v'_2 must be larger than $v'_1 = v_1$ and must be one of the values v_j (or 1), the only choice is to have $v'_2 = v_2$. Similarly, for $i = 3$, we have $v'_3 \le v_3$. Since v'_3 must be larger than $v'_1 = v_1$ and larger than $v'_2 = v_2$, and it must be one of the values v_j (or 1), the only choice is to have $v'_3 = v_3$.

In a similar manner, we can prove that $v'_i = v_i$ for all $i < k$, i.e., that

$$(\pi \mid \Psi)(\omega) = \pi(\omega) \text{ for all } \omega \text{ for which } \pi(\omega) < \max_{\omega' \in \Omega} \pi(\omega').$$

For the alternatives ω for which $\pi(\omega) = \max\limits_{\omega' \in \Omega} \pi(\omega') = v_k$, equal values of $\pi(\omega)$ must map into equal values of $(\pi \mid \Psi)(\omega)$. The corresponding value v'_k must be larger than $v'_{k-1} = v_{k-1}$, and it must be one of the values v_j or 1. So, we have either $v'_k = v_k$ or $v'_k = 1$.

In the first case, when $v'_k = v_k$, the largest value of $(\pi \mid \Psi)(\omega)$ is $v_k < 1$, which contradicts to the fact that, by definition of a conditioning operator, these values must form a possibility distribution. Thus, we must have $v'_k = 1$, i.e., we must have

$$(\pi \mid \Psi)(\omega) = 1 \text{ for all } \omega \text{ for which } \pi(\omega) = \max_{\omega' \in \Omega} \pi(\omega').$$

So, indeed, we have derive the min-based conditioning from the properties **C1**–**C6**. The proposition is proven.

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