Why superellipsoids: an explanation

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Need to describe uncertainty domains. The intent of the mass production of a gadget is to produce gadgets with identical values $(x_1, \ldots, x_n)$ of the desired characteristics x_i. In reality, of course, different gadgets end up having slightly different values $\widetilde{x}_i$ of these characteristics: $\Delta x_i \stackrel{\text{def}}{=} \widetilde{x}_i - x_i \neq 0$. For each of these characteristics x_i, we usually have a tolerance bound Δx_i for which $|\Delta x_i| \leq \Delta_i$, so that possible values of Δx_i form an interval $[-\Delta_i, \Delta_i]$. Thus, possible values of the deviation vector $\Delta x = (\Delta x_1, \ldots, \Delta x_n)$ are located in the box $[-\Delta_1, \Delta_1] \times \ldots \times [-\Delta_n, \Delta_n]$. In practice, not all vectors Δx from this box are possible. It is therefore desirable to describe the set of all possible deviation vectors Δx. This set is known as the *uncertainty domain.*

Shall not we also determine probabilities? At first glance, it seems that we should be interested not only in finding out which deviation vectors Δx are possible and which are not, but also in how frequent different possible vectors are. In other words, we should be interested not only in the uncertainty domain, but also on the probability distribution on this domain. In reality, however, it is not possible to find these probabilities. Indeed, the manufacturing process may slightly change (and often does change). After each such change, the tolerance intervals and the resulting uncertainty domain remain largely unchanged, but the probabilities change (often drastically).

Empirical shapes of uncertainty domains. Empirical analysis has shows that in many practical cases, the uncertainty domain can be well approximated by a *super-ellipsoid* $\sum\limits_{i=1}^{n}\left(\frac{|\Delta x_i|}{\sigma_i}\right)^p \le C$ for some values σ_i, p, and C, and the accuracy of this approximation is higher than for other approximation families with the same number of parameters.

What we do in this paper. In this paper, we provide a theoretical explanation for this empirical phenomenon.

Our idea. In reality, there is some probability distribution $\rho_i(\Delta x_i)$ for each of the random variables Δx_i. Since we have no reason to assume that positive values are more probable than negative values or vice versa, it makes sense to assume that they are equally probable, i.e., that each distribution $\rho_i(\Delta x_i)$ is symmetric: $\rho_i(\Delta x_i) = \rho_i(|\Delta x_i|)$. Similarly, since we have no reasons to believe that different deviations are statistically dependent, it makes sense to assume that the corresponding random variables are independent. In this case, the overall probability density function (pdf) has the form $\rho(\Delta x) = \prod\limits_{i=1}^{n} \rho_i(|\Delta x_i|)$.

Usually, we consider a deviation vector possible if its probability exceed a certain threshold t. Thus, the desired set has the form $S_t \stackrel{\text{def}}{=} \{\Delta x : \rho(\Delta x) \ge t\}$. Numerical values of the deviations Δx_i depend on the choice of a measuring unit; if we replace the original unit by a unit which is λ times smaller, then for the exact same physical situation, we get the new numerical values $\Delta x_i' = \lambda \cdot \Delta x_i$. Since the physics remains the same, it makes sense to require that the uncertainty domains do not change under such a re-scaling. To be more precise, the pdf threshold t may change, but the family of such sets should remain unchanged: $\{S_t'\}_t = \{S_t\}_t$, where S_t' corresponds to the re-scaled pdf $\rho'(\Delta x) = \text{const} \cdot \rho(\lambda \cdot \Delta)$.

We prove that under this scale-invariance, the corresponding sets S_t are exactly super-ellipsoids. Thus, we get the desired explanation.

References:

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