Decision Making Under Interval Uncertainty as a Natural Example of a Quandle

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Need for decision making. In many real-life problems, we need to select an alternative a from the list of possible alternatives – e.g., we want to select a design and/or location of a plant, a financial investment, etc. In many such situations, we have a well-defined objective function $u(a)$ that describes our preferences. If we know the exact value of $u(a)$ for each alternative a, then we select the alternative with the largest value of $u(a)$.

Decision making under interval uncertainty. In practice, we usually only know the consequences of each decision with some uncertainty. Often, the only information that we have about the corresponding values of $u(a)$ is that it is somewhere between the known bounds $\underline{u}(a)$ and $\overline{u}(a)$, i.e., that $u(a) \in [\underline{u}(a), \overline{u}(a)]$. In such situations, to make a definite decision, we need to assign, to each such interval, a single numerical value u_0 describing the quality of the corresponding alternative. We will denote this value u_0 by $\overline{u}(a) \triangleright \underline{u}(a)$.

Natural properties of the corresponding operation $\triangleright$. What are the natural properties of the operation $a \triangleright b$?

First, if we know the exact value of $u(a)$, i.e., if the corresponding interval has the form $[x, x]$ for some x, then the corresponding equivalent value is simply equal to x: $x \triangleright x = x$.

Another reasonable property is monotonicity: if $x < x'$, then $x \triangleright y > x' \triangleright y$. (For continuous functions, monotonicity is equivalent to invertibility of $x \to x \triangleright y$.)

To get the third property, let us consider a slightly more complex situation, when we know the lower bound z of the corresponding interval, but we do not know its upper bound: we only know that this upper bound is between y and x. We can analyze this situations in two different ways.

First, we can say that since all we know about the upper bound is that it is between y and x, this upper bound is therefore equivalent to the value $y \rhd x$. Now, after we have thus reduced the uncertain upper bound to a single number, the original information becomes simply an interval with an exact lower bound z and an exact upper bound $x \rhd y$. We can now apply the operation $\rhd$ to estimate the equivalent value of this interval as $(x \rhd y) \rhd z$.

There is also an alternative approach. For each possible value v between y and x, we have an interval $[z, v]$ with equivalent value $v \rhd z$. Due to the natural monotonicity, this equivalent value is the smallest when v is the smallest, i.e., when $v = y$, and it is the largest when v is the largest, i.e., when $v = x$. Thus, possible equivalent values form an interval $[y \rhd z, x \rhd z]$. The equivalent value of this interval is therefore $(x \rhd z) \rhd (y \rhd z)$.

It is reasonable to require that these two approaches lead to the same value, i.e., that $(x \rhd y) \rhd z = (x \rhd z) \rhd (y \rhd z)$.

Hurwicz optimism-pessimism criterion satisfies these properties. One can easily check that the widely used Hurwicz criterion $x \rhd y = \alpha \cdot x + (1 - \alpha) \cdot y$, with $\alpha > 0$, satisfies the above properties.

This is a quandle. Interestingly, the above three natural properties are well known in knot theory: sets with operations satisfying these properties are knows as *quandles* [1]. Since decision making under interval uncertainty naturally leads to a quandle, maybe we will be able to apply results from quandle theory to make better decisions?

References:

[1] M. Elhamdadi, S. Nelson, *Quandles: An Introduction to the Algebra of Knots*, American Mathematical Society, Providence, Rhode Island, 2015.