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Introduction

According to the decision theory, a reasonable person should select an alternative \( a \) for which an appropriate objective function \( u(a) \) – called utility – attains its largest possible value. The utility function is usually selected in such a way that if for some action \( a \), we know the probabilities \( p_i \) of different outcomes \( o_i \), then the utility of \( a \) is equal to the expected value of the utilities:

\[
u(a) = \sum_{i=1}^{n} p_i \cdot u(o_i).
\]

Such a utility function is determined uniquely modulo a linear transformation \( u(a) \rightarrow u'(a) = k \cdot u(a) + \ell \), where \( k > 0 \).

For some actions, we have no information about the probabilities of different outcomes \( o_i \). In this case, all we know about the expected utility \( u(a) \) is that it is in the interval \([u^-(a), u^+(a)]\), where \( u^-(a) = \min u(o_i) \) and \( u^+(a) = \max u(o_i) \). To make decisions under such interval uncertainty, we must, in particular, be able to compare such actions with actions for which we know the expected utility \( u(a) \). Thus, we need to be able to assign, to each interval \([u^-(a), u^+(a)]\), an equivalent utility value \( u(a) \).

A way to assign such an equivalent utility value was proposed by a Nobel Prize winner Leo Hurwicz:

\[
u(a) = \alpha \cdot u^+(a) + (1 - \alpha) \cdot u^-(a),
\]

where \( \alpha \in [0, 1] \) describes the optimism level of the decision maker: \( \alpha = 1 \) means that the decision maker only takes into account the best-case scenario, \( \alpha = 0 \) means that only the worst-case scenario is taken into account, and \( \alpha \in (0, 1) \) means that both best-case and worst-case scenarios are taken into account. It turns out that the Hurwicz assignment is invariant relative to linear transformations of utility – and it is the only invariant assignment.

In practice, sometimes, we do not know the exact values of \( u(a) \) and \( \bar{u}(a) \). For example, we may only know the bounds on each of these bounds: we know that \( u(a) \in [u^-(a), u^+(a)] \) and that \( \bar{u}(a) \in [\bar{u}^-(a), \bar{u}^+(a)] \). Such a situation is known as a twin interval. How can we make decisions under such twin interval uncertainty?

General Idea

Our main idea is to use Hurwicz assignment several times. Specifically, for the lower bound \( u(a) \), all we know that it is in the interval \([u^-(a), u^+(a)]\). According to the Hurwicz assignment, this is equivalent to having \( u(a) = \alpha \cdot u^+(a) + (1 - \alpha) \cdot u^-(a) \).

Similar, by applying the Hurwicz assignment to the interval \([\bar{u}^-(a), \bar{u}^+(a)]\), we conclude that the upper bound is equivalent to \( \bar{u}(a) = \alpha \cdot \bar{u}^-(a) + (1 - \alpha) \cdot \bar{u}^+(a) \).
Thus, the original twin interval is equivalent to the interval \([u(a), \overline{u}(a)]\), for which the Hurwicz assignment produces an equivalent value
\[
    u(a) = \alpha \cdot \overline{u}(a) + (1 - \alpha) \cdot u(a) = \\
    \alpha^2 \cdot \overline{u}^+(a) + \alpha \cdot (1 - \alpha) \cdot \overline{u}^-(a) + \alpha \cdot (1 - \alpha) \cdot u^+(a) + (1 - \alpha)^2 \cdot u^-(a).
\]

Alternatively, we can consider the situation differently: we do not consider the actual interval. The smallest possible interval – in terms of component-wise order – is \([u^-(a), \overline{u}^-(a)]\). The largest possible interval is \([u^+(a), \overline{u}^+(a)]\). For the smallest interval, Hurwicz’s equivalent utility is
\[
    u^-(a) = \alpha \cdot \overline{u}^-(a) + (1 - \alpha) \cdot u^-(a).
\]
For the largest interval, the equivalent utility is \(u^+(a) = \alpha \cdot \overline{u}^+(a) + (1 - \alpha) \cdot u^+(a)\). Thus, possible values of utility form an interval \([u^-(a), u^+(a)]\). For this interval, the Hurwicz equivalent value is \(\alpha \cdot u^+(a) + (1 - \alpha) \cdot u^-(a)\), which is, as one can check, exactly equal to the above value.

Applications

Some physical quantities we can measure directly. However, in many practical situations, we are interested in a quantity which is difficult (or even impossible) to measure directly. To estimate the values of such a quantity, a natural idea is to find auxiliary easier-to-measure quantities \(x_1, \ldots, x_n\) which are related to \(y\) by a known dependence \(y = f(x_1, \ldots, x_n)\), and then use the results \(\tilde{x}_i\) of measuring \(x_i\) to compute the estimate \(\tilde{y} = f(\tilde{x}_1, \ldots, \tilde{x}_n)\).

Often, the only information that we have about each measurement error \(\Delta x_i \overset{\text{def}}{=} \tilde{x}_i - x_i\) is the upper bound \(\Delta_i\), on its absolute value: \(|\Delta x_i| \leq \Delta_i\). In this case, the only information that we have about the actual (unknown) value \(x_i\) is that \(x_i\) belongs to the interval \([\underline{x}_i, \overline{x}_i] = [\tilde{x}_i - \Delta_i, \tilde{x}_i + \Delta_i]\). Usually, we do not know the dependence between the values \(x_i\) (and we do not even know if there is a dependence).

The traditional interval approach to this situation is to conclude that \(y\) belongs to the range \(y = \{f(x_1, \ldots, x_n) : x_i \in [\underline{x}_i, \overline{x}_i]\}\). However, in reality, the range \([\underline{y}, \overline{y}]\) depends on the possible dependence between the variables \(x_i\). In general, \(y = \inf \{f(x_1, \ldots, x_n) : (x_1, \ldots, x_n) \in S\}\) and \(\overline{y} = \sup \{f(x_1, \ldots, x_n) : (x_1, \ldots, x_n) \in S\}\), where \(S\) is a set for which, for every \(i\), the projection \(\pi_i(S)\) on the \(i\)-th axis coincides with \([\underline{x}_i, \overline{x}_i]\).

For different sets \(S\), we have different values \(\overline{y}\) and \(\underline{y}\). It is therefore desirable to compute the ranges \([\underline{y}^-, \overline{y}^+]\) and \([\underline{y}^-, \overline{y}^+]\) of the corresponding values – i.e., to compute the corresponding twin interval. Here, \(\underline{y}^\perp\) and \(\overline{y}^\perp\) are the endpoints of the range \(y\), which can be computed by the usual interval techniques, so the question is how to compute \(\overline{y}^\perp\) and \(\underline{y}^\perp\).

In the talk, we show how to compute these bounds for the practically important case when quadratic and higher order terms in \(\Delta x_i\) can be safely ignored, and thus, \(\Delta y = \overline{y} - \underline{y} = \sum\limits_{i=1}^{n} c_i \cdot \Delta x_i\), where \(c_i = \frac{\partial f}{\partial x_i}(\tilde{x}_1, \ldots, \tilde{x}_n)\). In turns out that in this case,
\[
\underline{y}^\perp = \overline{y} - 2 \max_i (|c_i| \cdot \Delta_i) - \sum\limits_{i=1}^{n} (|c_i| \cdot \Delta_i)\quad \text{and} \quad \overline{y}^\perp = \overline{y} - 2 \max_i (|c_i| \cdot \Delta_i) + \sum\limits_{i=1}^{n} (|c_i| \cdot \Delta_i).
\]
In particular, for arithmetic operations like \(f(x_1, x_2) = x_1 + x_2\), this means that the sum, product, etc., of two intervals is now viewed as a twin interval.

We can then use the above formulas for decision making under twin interval uncertainty to make appropriate decisions.