Adjoint Fuzzy Partition and Generalized Sampling Theorem

Irina Perfilieva[a], Michal Holčapek[a], and Vladik Kreinovich[b]

[a]University of Ostrava,
Institute for Research and Applications of Fuzzy Modeling,
Centre of Excellence IT4Innovations,
30. dubna 22, 701 03 Ostrava 1, Czech Republic

[b]Department of Computer Science, University of Texas at El Paso
500 W. University, El Paso, TX 79968, USA

E-mails: Irina.Perfilieva,Michal.Holcapek@osu.cz, vladik@utep.edu

Abstract. A new notion of adjoint fuzzy partition is introduced and the reconstruction of a function from its F-transform components is analyzed. An analogy with the Nyquist-Shannon-Kotelnikov sampling theorem is discussed.

Keywords: F-transform, adjoint fuzzy partition, sampling theorem, Nyquist-Shannon-Kotelnikov reconstruction

1 Introduction

We analyze the problem of whether a function can be reconstructed from a countable set of its F-transform components. We prove that if a function fulfills the same conditions as in the Nyquist-Shannon-Kotelnikov theorem (also known as a sampling theorem), see [4, 6, 12], then the above mentioned reconstruction is possible and moreover, the sampling theorem is its particular case.

Our inspiration came from the following analogy: similar to the F-transform components, signal samples can be computed on the basis of the partition generated by Dirac's delta function δ. On the other hand, the reconstruction is performed with the help of another partition generated by the function sinc. We analyzed the interconnection between δ and sinc and extracted a principal characteristic that we call *adjointness*. If partitions are generated by adjoint functions, they are called *adjoint* as well. Adjoint fuzzy partitions are used in the direct and newly defined inverse F-transform so that their mutually inverse correspondence is guaranteed for functions that fulfill the same conditions as in the standard sampling theorem.

The F-transform is very useful in many applications such as image and signal processing, image compression, time series prediction, etc.; see, e.g., [2, 5, 8, 9]. The initially proposed inverse F-transform [8] is lossy; i.e., except for constant functions, it produces a result that is different from an original object. This fact motivated us to modify the definition of the inverse F-transform to extend the space of original functions, for which direct and inverse F-transforms are mutually inverse.

In the proposed contribution$^{\star)}$, we give a short overview of the F-transform theory and its evolution. We discuss various fuzzy partitions and extend the notion of the inverse F-transform. We introduce a notion of an adjoint fuzzy partition and discuss its properties. Finally, we prove the main theoretical result about reconstruction from a countable set of F-transform components.

2 Preliminaries: Nyquist-Shannon-Kotelnikov Reconstruction

In this section, we provide a short review of the background of the sample-based reconstruction of a band-limited signal.

We assume that a digital signal is identified with a function varying in time, which is assumed to have a Fourier transform that is zero outside some bounded interval (in other words, a signal is *band-limited* to a given *bandwidth*). The sampling theorem (also known as Nyquist-Shannon-Kotelnikov theorem, see [4, 6, 12]) characterizes what is sufficient for full reconstruction of a signal from a set of its samples.

Theorem 1 (Sampling Theorem).
Let $x \in L_2(\mathbb{R})$ be continuous and band-limited, i.e., $\widehat{x}(\omega) = 0$ for $|\omega| > \Omega$ where $\widehat{x}$ is the Fourier transform of x and Ω is some positive constant. Then, x can be determined by its values at a discrete set of points:

$$x(t) = \sum_{k=-\infty}^{\infty} x\left(\frac{k\pi}{\Omega}\right) \cdot \frac{\sin(\Omega t - k\pi)}{\Omega t - k\pi}. \tag{1}$$

We will be using the following notation: $h = \frac{\pi}{\Omega}$, $t_k = \frac{k\pi}{\Omega} = k \cdot h$ and the corresponding reconstruction formula:

$$x(t) = \sum_{k=-\infty}^{\infty} x(t_k) \cdot \operatorname{sinc}\left(\frac{t}{h} - k\right), \tag{2}$$

where

$$\operatorname{sinc}(t) \stackrel{\text{def}}{=} \frac{\sin(\pi t)}{\pi t}.$$

3 The F-Transform: Short Overview and Evolution

The F-transform (originally, *fuzzy transform*) is a particular integral transform whose peculiarity consists in using a *fuzzy partition* of a universe of discourse

$^{\star)}$ The extended version of this contribution together with the application to the problem of function “de-noising” was submitted to [11].

(usually, $\mathbb{R}$). We observe that the F-transform method was motivated by the ideas and techniques of fuzzy logic (see, e.g., [15]) and especially by the Takagi-Sugeno models [14]. In addition, the idea of a fuzzy partition was derived from observing a collection of antecedents in a fuzzy rule based system. The direct F-transform components are possible consequents in the Takagi-Sugeno model with singletons.

The F-transform has two phases: direct and inverse (see details in [8]). The direct F-transform is applied to functions from $L_2(\mathbb{R})$ and maps them linearly onto sequences (originally finite) of numeric/functional components. The inverse F-transform smoothly approximates the original function.

Let us remark that almost all fuzzy approximation models, including Takagi-Sugeno models [14], are based on linear-like combinations of fuzzy sets with numeric or functional coefficients. The principal difference between them and the inverse F-transform is in the computation of coefficients. In the F-transform case, these coefficients are weighted orthogonal projections on subdomains, such that the best approximation in a local sense is guaranteed. In Takagi-Sugeno models, the coefficients guarantee that the corresponding approximating function is a best approximation on a whole domain in the sense of the L_2 metric. Similar models have been considered in [1, 7].

3.1 Fuzzy partition

The notion of a fuzzy partition does not have a nonambiguous meaning in fuzzy literature. We will not go into full detail but concentrate on an evolution of this notion in connection with the F-transform (see [3, 10, 13]).

A *fuzzy partition with the Ruspini condition* was introduced in [8] as a collection of bell-shaped fuzzy sets $A_1, \dots, A_n$ on the real interval $[a, b]$ with continuous membership functions, such that for all $x \in [a, b]$,

$$\sum_{k=1}^{n} A_k(x) = 1.$$

This partition can be characterized as a "partition-of-unity".

In [10], a generalized fuzzy partition without the Ruspini condition was proposed with the purpose of obtaining a better approximation by the inverse F-transform.

Below, in Definition 1, we introduce a particular case of a generalized fuzzy partition that is determined by a generating function. We say that function $a : \mathbb{R} \longrightarrow [0, 1]$ is a *generating function of a fuzzy partition* (a *generating function*, for short), if it is non-negative, continuous, even, bell-shaped and moreover, it vanishes outside $[-1, 1]$ and fulfills $\int_{-1}^{1} a(t)\, dt = 1$. Below, we give the example of a generating function, which we call the *raised cosine*:

$$a^{cos}(t) = \begin{cases} \frac{1}{2}(1 + \cos(\pi t)), & -1 \le t \le 1, \\ 0, & \text{otherwise.} \end{cases} \quad (3)$$

Generating function a produces infinitely many *rescaled* functions $a_H : \mathbb{R} \longrightarrow [0,1]$ such that

$$a_H(t) \stackrel{\text{def}}{=} a\left(\frac{t}{H}\right),$$

where H is a positive number called a *scale factor*.

Definition 1. *Let $a : \mathbb{R} \longrightarrow [0,1]$ be a generating function of a fuzzy partition, i.e., a is non-negative, continuous, even, bell-shaped, vanishes outside $[-1,1]$ and fulfills $\int_{-1}^{1} a(t)\,dt = 1$. Let $h > 0$, $t_k = t_0 + k \cdot h$, $k \in \mathbb{Z}$, be uniformly distributed nodes$^{\star\star)}$ in $\mathbb{R}$. Let $H > \frac{h}{2}$ and a_H be an H-rescaled version of a. With each node t_k, we correspond the translation $a_k(t) = a_H(t_k - t)$. We say that the set $\{a_k,\, k \in \mathbb{Z}\}$ establishes an (h,H)-uniform fuzzy partition of $\mathbb{R}$. Functions a_k are called basic functions.*

By the condition $H > \frac{h}{2}$, each point from $\mathbb{R}$ is "*covered*" by at least one basic function - by this we mean that the value of this function at this point is greater than zero. By the condition $h > 0$, each point from $\mathbb{R}$ is covered by at most a finite number of basic functions.

It is easy to see that (substituting $s = \frac{t}{H}$)

$$\int_{-\infty}^{\infty} a_H(t)\,dt = \int_{-H}^{H} a_H(t)\,dt = \int_{-H}^{H} a\left(\frac{t}{H}\right)\,dt = H \cdot \int_{-1}^{1} a(s)\,ds = H. \tag{4}$$

If $h = H$, then an (h,H)-uniform fuzzy partition is called an *h-uniform fuzzy partition.*

The following lemma will be used in the sequel.

Lemma 1. *Let $a : \mathbb{R} \longrightarrow [0,1]$ be a generating function so that it is continuous, even, bell-shaped, vanishes outside $[-1,1]$ and fulfills $\int_{-1}^{1} a(t)\,dt = 1$. Then, the following is valid:*

$$\frac{1}{2} \leq \|a\|^2 \leq 1, \tag{5}$$

where $\|a\|$ is the norm in $L_2([-1,1])$.

In particular, if $a = a^{cos}$, then $\|a^{cos}\|^2 = \frac{3}{4}$.

3.2 Direct and Inverse F-transform

In this section, we review formal notions of the direct and inverse F-transforms as introduced in [8] and extend the latter.

Assume that $x \in L_2(\mathbb{R})$ and $\{a_k,\, k \in \mathbb{Z}\}$ is an (h,H)-uniform fuzzy partition of $\mathbb{R}$, where $a_k(t) = a_H(t_k - t)$, a_H is the H-rescaled generating function a, and $t_k = k \cdot h$, $k \in \mathbb{Z}$, are nodes. The sequence $F[x] = \{X_k,\, k \in \mathbb{Z}\}$, where

$$X_k = \frac{\int_{-\infty}^{\infty} a_k(s) \cdot x(s)\,ds}{\int_{-\infty}^{\infty} a_k(s)\,ds}, \tag{6}$$

$^{\star\star)}$ For simplicity of representation, we assume that $t_0 = 0$.

is called the *(direct) F-transform* of x with respect to $\{a_k,\, k \in \mathbb{Z}\}$. Real numbers $X_k, k \in Z$, are called the *F-transform components* of x. Due to the assumption of uniformity of the partition and by (4), the representation (6) of X_k can be simplified as follows:

$$X_k = \frac{\int_{-\infty}^{\infty} a_H(t_k - s) \cdot x(s)\, ds}{\int_{-\infty}^{\infty} a_H(t_k - s)\, ds} = \frac{1}{H} \int_{-\infty}^{\infty} a_H(t_k - s) \cdot x(s)\, ds. \quad (7)$$

It is easy to see that if $x, y \in L_2(\mathbb{R})$, $\alpha \in \mathbb{R}$, then

$$\begin{aligned} F[x+y] &= F[x] + F[y], \\ F[\alpha x] &= \alpha F[x]. \end{aligned} \quad (8)$$

The basic idea of the F-transform is to "capture" a local behavior of an original function and characterize it by a certain value. It follows from (6) that the F-transform can be effectively computed for a rather wide class of functions. In particular, all continuous functions on compact domains can be originals of the F-transform.

Let $\mathbf{x} = (X_k,\, k \in \mathbb{Z})$ be an arbitrary sequence of reals and $\{a_k,\, k \in \mathbb{Z}\}$ be an (h, H)-uniform fuzzy partition of $\mathbb{R}$ with the H-rescaled generating function a. The following *inversion formula*

$$\hat{\mathbf{x}}^F(t) = \frac{\sum_{k=-\infty}^{\infty} X_k \cdot a_k(t)}{\sum_{k=-\infty}^{\infty} a_k(t)},\; t \in \mathbb{R}, \quad (9)$$

converts the sequence $\mathbf{x}$ into the real valued function $\hat{\mathbf{x}}^F$. Because the parameter h in an (h, H)-uniform fuzzy partition $\{a_k,\, k \in \mathbb{Z}\}$ of $\mathbb{R}$ is greater than zero, both sums in (9) contain only a finite number of non-zero summands. Because $H > \frac{h}{2}$, each point from $\mathbb{R}$ is covered by at least one basic function, so that the denominator in (9) is always non-zero. Therefore, the expression in (9) is well defined.

We say that the function $\hat{\mathbf{x}}^F$ is the *inverse F-transform of the sequence* $\mathbf{x} = (X_k,\, k \in \mathbb{Z})$ with respect to the fuzzy partition $\{a_k,\, k \in \mathbb{Z}\}$. If the sequence $\mathbf{x}$ consists of the F-transform components of some function x with respect to $\{a_k,\, k \in \mathbb{Z}\}$, then $\hat{\mathbf{x}}^F$ is simply called the *inverse F-transform* of x.

The inverse F-transform $\hat{\mathbf{x}}^F$ of a continuous function x can approximate x with an arbitrary precision. The desired quality of approximation can be achieved by a special choice of a partition. This fact can be easily proved using the technique introduced in [8].

4 Reconstruction from the F-transform Components

The F-transform is the result of a linear correspondence between a set of functions from $L_2(\mathbb{R})$ and a set of sequences of reals. In general, the inversion formula does not define the inverse correspondence. In [8], it has been shown that the inverse F-transform can approximate a continuous function with an arbitrary precision. In the later publications [1, 7], other smooth approximations for functions from $L_2(\mathbb{R})$ by the inverse F-transforms were proposed.

Below, we show even more; namely, the original function can be reconstructed from its F-transform components. Of course, this result can be established for a narrower than $L_2(\mathbb{R})$ class of functions. Our motivation stems from the Nyquist-Shannon-Kotelnikov reconstruction theorem discussed above.

4.1 Adjoint partition

If a fuzzy partition is fixed, then both direct and inverse F-transforms are uniquely determined by this partition. If we require the inverse F-transform to be coincident with the original function, we shall change its main parameter – the fuzzy partition.

Definition 2. *Let $\{a_k,\ k \in \mathbb{Z}\}$ be an (h, H)-uniform fuzzy partition of $\mathbb{R}$, where $a_k(t) = a_H(t_k - t)$, a_H is the H-rescaled generating function a and $t_k = k \cdot h,\ k \in \mathbb{Z}$, are uniformly distributed nodes. We say that the set of functions $\{b_k,\ k \in \mathbb{Z}\}$, establishes an adjoint (h, H)-*uniform partition *of $\mathbb{R}$ (with respect to $\{a_k,\ k \in \mathbb{Z}\}$), if $b_k(t) = b_H(t - t_k)$ are translations of the continuous function $b_H : \mathbb{R} \longrightarrow \mathbb{R}$ with the same nodes $t_k,\ k \in \mathbb{Z}$, and b_H is determined by*

$$\widehat{a_H} \cdot \widehat{b_H} = \mathbf{1}_{[-\Omega,\Omega]}, \tag{10}$$

where $\Omega > 0$ is some positive constant, $\mathbf{1}_{[-\Omega,\Omega]}$ is a characteristic function of $[-\Omega, \Omega]$ and $\widehat{a_H}$, $\widehat{b_H}$ are the Fourier transforms of a_H and b_H, respectively.

The lemma given below gives a necessary and sufficient condition on an $(h, 1)$-uniform fuzzy partition that guarantees the existence of the adjoint one.

Lemma 2. *Let $\{a_k,\ k \in \mathbb{Z}\}$, be an $(h, 1)$-uniform fuzzy partition of $\mathbb{R}$ with generating function $a : \mathbb{R} \longrightarrow [0, 1]$, such that $a_k(t) = a(t - t_k)$ and $t_k = k \cdot h,\ k \in \mathbb{Z}$, are nodes. Then, the adjoint partition $\{b_k,\ k \in \mathbb{Z}\}$ exists if and only if there exists $\Omega > 0$ such that for all $\omega \in [-\Omega, \Omega]$,*

$$\widehat{a}(\omega) \neq 0. \tag{11}$$

Moreover, the adjoint partition $\{b_k,\ k \in \mathbb{Z}\}$ is determined by h-translations of function $b : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$b(t) = \frac{1}{2\pi} \int_{-\Omega}^{\Omega} \frac{e^{i\omega t}}{\widehat{a}(\omega)}\, d\omega. \tag{12}$$

Remark 1. Let $\{a_k,\ k \in \mathbb{Z}\}$ be an (h, H)-uniform fuzzy partition of $\mathbb{R}$, where $a_k(t) = a_H(t_k - t)$ and a_H is the H-rescaled generating function a. Let $\{b_k,\ k \in \mathbb{Z}\}$, where $b_k(t) = b_H(t - t_k)$ be the adjoint (h, H)-uniform partition of $\mathbb{R}$ with respect to $\{a_k,\ k \in \mathbb{Z}\}$.

In Remark 1, we discuss some particular properties of functions $b_k,\ k \in \mathbb{Z}$.

(i) The function b_H is a rescaled version of a certain function $b : \mathbb{R} \longrightarrow \mathbb{R}$ in both vertical and horizontal directions. Specifically,

$$b_H(t) = \frac{1}{H^2} \cdot b\left(\frac{t}{H}\right), \tag{13}$$

where b is determined as follows:

$$\widehat{a} \cdot \widehat{b} = \mathbf{1}_{[-H\Omega, H\Omega]}. \tag{14}$$

Indeed, equality (13) easily follows from (14) and the scaling property of the Fourier transform applied to the function a:

$$\widehat{a_H}(\omega) = H\widehat{a}(H\omega).$$

(ii) The explicit representation of a particular function b_k, $k \in \mathbb{Z}$ as a translation and rescaling of the function b is as follows:

$$b_k(t) = b_H(t - t_k) = \frac{1}{H^2} \cdot b\left(\frac{t - t_k}{H}\right). \tag{15}$$

This representation justifies the name "partition", assigned to the set $\{b_k,\, k \in \mathbb{Z}\}$. Moreover, as we see in Lemma 3 below, the generating function b fulfills the extended Ruspini condition (16).

We call b a *generating function of the adjoint* (h, H)*-uniform partition* $\{b_k,\, k \in \mathbb{Z}\}$,$^{\star\star\star)}$, which corresponds to the (h, H)-uniform fuzzy partition $\{a_k,\, k \in \mathbb{Z}\}$, determined by a. If $h = H$, we simply call both partitions as h-uniform.

As the following result shows, the set of translations (without rescaling) of a generating function of an adjoint H-uniform partition establishes the Ruspini partition. This is an additional argument in favor of using the word "partition" in the notion of adjoint partition.

Lemma 3. *Let* $a : \mathbb{R} \longrightarrow [0, 1]$ *be a generating function such that for all* $\omega \in [-\Omega, \Omega]$, $\widehat{a}(\omega) \neq 0$, *where* $\Omega > 0$ *is some positive constant. Let* $H = \frac{\pi}{\Omega}$ *and* $\{a_k,\, k \in \mathbb{Z}\}$, *be an* H*-uniform fuzzy partition such that* $a_k(t) = a_H(t - t_k)$, a_H *is the* H*-rescaled generating function* a *and* $t_k = k \cdot H$, $k \in \mathbb{Z}$. *Let* $\{b_k,\, k \in \mathbb{Z}\}$, *where* $b_k(t) = b_H(t - t_k)$, *be the adjoint* H*-uniform partition of* $\mathbb{R}$ *with respect*

$^{\star\star\star)}$ We distinguish between a generating function of an adjoint partition (in this paper, denoted by b) and a generating function of a fuzzy partition (in this paper, denoted by a). The latter is characterized in Definition 1, while the former is associated with an adjoint partition and can have values outside the interval $[0, 1]$.

to $\{a_k,\, k \in \mathbb{Z}\}$ *with the generating function* b*. Then, for all* $t \in \mathbb{R}$*,*

$$\sum_{k=-\infty}^{\infty} b\left(\frac{t}{H} - k\right) = 1, \tag{16}$$

$$\sum_{k=-\infty}^{\infty} b_k(t) = \frac{1}{H^2}, \tag{17}$$

$$\sum_{k=-\infty}^{\infty} b^2\left(\frac{t}{H} - k\right) = \|b\|^2 < \infty, \tag{18}$$

where $\|\cdot\|$ *is the norm in* $L_2(\mathbb{R})$*.*

At the end of this subsection, we give a particular example of an h-uniform partition of $\mathbb{R}$ and its adjoint where the latter has an analytic representation.

Example 1. We consider an h-uniform partition $\{\delta_k,\, k \in \mathbb{Z}\}$ of $\mathbb{R}$, where $\delta_k(t) = \delta(t - t_k)$, $t_k = k \cdot h$ and δ is the Dirac's delta function[†)]. Although this partition is not fuzzy (it is generated by the non-bounded delta function), it fulfills all the assumptions of Lemma 2, including the main condition (11). The latter is because for all $\omega \in \mathbb{R}$, $\widehat{\delta}(\omega) = 1$, so that we can choose an arbitrary bounded interval $[-\Omega, \Omega]$ where this condition is fulfilled. We choose $\Omega = \pi$ and apply the proof of Lemma 2 to the partition $\{\delta_k,\, k \in \mathbb{Z}\}$. After substitution into (12), we easily obtain the generating function sinc of the adjoint to $\{\delta_k,\, k \in \mathbb{Z}\}$ partition, so that

$$b(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\omega t} d\omega = \frac{1}{\pi t} \sin(\pi t) = \mathrm{sinc}(t). \tag{19}$$

The resulting adjoint h-uniform partition is given by the set of functions $\{\mathrm{sinc}_k,\, k \in \mathbb{Z}\}$, where $\mathrm{sinc}_k(t) = \mathrm{sinc}(t - t_k)$, so that sinc is its generating function.

In Figure 1, we demonstrate graphs of generating functions of the two adjoint uniform partitions of $\mathbb{R}$ with respect to two uniform partitions with the following generating functions: δ (Dirac's delta) and a^{cos} (raised cosine). The latter is given by (3), and it is of the fuzzy type.

In almost all cases, a computation of a generating function b of an adjoint partition cannot be performed analytically. It is a matter of a numeric computation on the basis of the expression (12). The example given Figure 1, has been numerically computed as well.

4.2 Main result

In this subsection, we show that a function that fulfills the same conditions as in the Nyquist-Shannon-Kotelnikov theorem (also known as a sampling theo-

[†)] Strictly speaking, the Dirac's delta is not a function, but a generalized function or a linear functional. Therefore, it makes sense to use it only if it appears inside an integral. In our paper, we always follow this restriction.

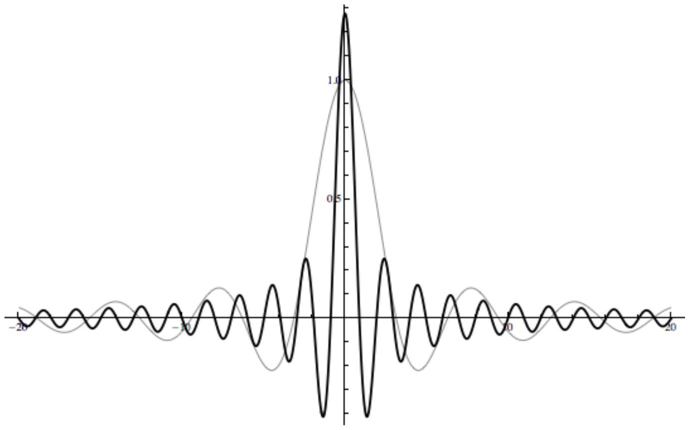

Fig. 1. Generating functions of the two adjoint uniform partitions of $\mathbb{R}$ with respect to uniform partitions with generating functions δ (in gray) and the raised cosine a^{cos} (in black).

rem) can be reconstructed from a countable set of its F-transform components. Moreover, we obtain the sampling theorem as a particular case.

Theorem 2 (Reconstruction from the F-transform).
Let function $x \in L_2(\mathbb{R})$ be continuous and band-limited, i.e., $\widehat{x}(\omega) = 0$ for $|\omega| > \Omega$, where Ω is some positive constant. Let $h = \frac{\pi}{\Omega}$, $H > h/2$ and a_H an H-rescaled version of the generating function a, such that for all $\omega \in [-\Omega, \Omega]$, $\widehat{a_H}(\omega) \neq 0$.

Let $\{b_k,\, k \in \mathbb{Z}\}$ be the adjoint (h, H)-uniform partition of $\mathbb{R}$ with respect to that given by $\{a_k,\, k \in \mathbb{Z}\}$, where $a_k(s) = a_H(t_k - s)$ and $t_k = k \cdot h$, $k \in \mathbb{Z}$.

Finally, let the sequence $\{X_k,\, k \in \mathbb{Z}\}$ consist of the F-transform components of x with respect to the fuzzy partition $\{a_k,\, k \in \mathbb{Z}\}$.

Then, x can be uniquely determined by its F-transform components, so that the following representation holds:

$$x(t) = \frac{H\pi}{\Omega} \sum_{k=-\infty}^{\infty} X_k \cdot b_k(t). \tag{20}$$

Proof. Let the assumptions be fulfilled, and consider the real function $X : \mathbb{R} \longrightarrow \mathbb{R}$, represented by the expression similar to (7)

$$X(t) = \frac{\int_{-\infty}^{\infty} a_H(t-s) \cdot x(s)\, ds}{H},\ t \in \mathbb{R}.$$

At the fixed nodes $t_k = k \cdot h$, $k \in \mathbb{Z}$, the values of X coincide with the corresponding F-transform components of x, i.e., $X(t_k) = X_k$, $k \in \mathbb{Z}$. We observe that the function X can also be represented by

$$X = \frac{1}{H}(a_H * x),$$

where $a_H * x$ is a convolution of a_H and x. We observe that $(a_H * x) \in L_2(\mathbb{R})$ and thus, $X \in L_2(\mathbb{R})$. Moreover, X is continuous. Therefore, by the properties of the Fourier transform, X can be represented by the inversion formula

$$X(t) = \underset{n\to\infty}{\mathrm{l.\,i.\,m.}}\ \frac{1}{2\pi}\int_{-n}^{n} \widehat{X}(\omega)e^{i\omega t}d\omega,\ ^{\ddagger)} \tag{21}$$

where (by convolution-to-product theorems)

$$\widehat{X}(\omega) = \frac{\widehat{x}(\omega)\cdot\widehat{a_H}(\omega)}{H}. \tag{22}$$

It follows that $\widehat{X}$ is band-limited and $\widehat{X}(\omega) = 0$ for $|\omega| > \Omega$. Therefore, by (21) and continuity of X, we have the exact representation

$$X(t) = \frac{1}{2\pi}\int_{-\Omega}^{\Omega} \widehat{X}(\omega)e^{i\omega t}d\omega. \tag{23}$$

Because $\widehat{X} \in L_2[-\Omega, \Omega]$, it can be expanded in a Fourier series

$$\widehat{X}(\omega) = \sum_{k=-\infty}^{\infty} g_k e^{-ik\pi\omega/\Omega} \tag{24}$$

where

$$g_k = \frac{1}{2\Omega}\int_{-\Omega}^{\Omega} \widehat{X}(\omega)e^{ik\pi\omega/\Omega}d\omega.$$

By (23),

$$g_k = \frac{\pi}{\Omega}X\left(\frac{k\pi}{\Omega}\right) = \frac{\pi}{\Omega}X(t_k) = \frac{\pi}{\Omega}X_k,$$

where X_k is the F-transform component of X with respect to $\{a_k,\, k \in \mathbb{Z}\}$. Substituting g_k into (24), we get

$$\widehat{X}(\omega) = \frac{\pi}{\Omega}\sum_{k=-\infty}^{\infty} X_k e^{-it_k\omega}. \tag{25}$$

Because the function x fulfills the assumptions of Theorem 1, we can express it with the help of the Fourier inversion formula

$$x(t) = \frac{1}{2\pi}\int_{-\Omega}^{\Omega} \widehat{x}(\omega)e^{i\omega t}d\omega,$$

‡) Here "l. i. m." indicates that the convergence is in $L_2(\mathbb{R})$ or in the quadratic mean sense.

where by (22),

$$\widehat{x}(\omega) = \frac{H \cdot \widehat{X}(\omega)}{\widehat{a_H}(\omega)}.$$

Hence, we have

$$x(t) = \frac{H}{2\pi} \int_{-\Omega}^{\Omega} \frac{\widehat{X}(\omega)}{\widehat{a_H}(\omega)} e^{i\omega t} d\omega,$$

and after substituting $\widehat{X}(\omega)$ from (25)

$$x(t) = \frac{H}{2\Omega} \sum_{k=-\infty}^{\infty} X_k \cdot \int_{-\Omega}^{\Omega} \frac{1}{\widehat{a_H}(\omega)} e^{i\omega(t-t_k)} d\omega. \tag{26}$$

Because $\widehat{a_H}$ is continuous (as the inverse Fourier transform of a function from $L_1(\mathbb{R})$) and $\widehat{a_H}(\omega) \neq 0$ in $[-\Omega, \Omega]$, the integral in the right-hand side of (26) exists for all $k \in \mathbb{Z}$. Therefore, equality (26) proves that x can be determined by the set of F-transform components.

To prove (20), we observe that by Lemma 2, the (h, H)-uniform fuzzy partition $\{a_k,\, k \in \mathbb{Z}\}$, where $a_k(s) = a_H(t_k - s)$, has the adjoint (h, H)-uniform partition $\{b_k,\, k \in \mathbb{Z}\}$ such that $b_k(t) = b_H(t - t_k)$ and

$$b_H(t) = \frac{1}{2\pi} \int_{-\Omega}^{\Omega} \frac{e^{i\omega t}}{\widehat{a_H}(\omega)}\, d\omega.$$

Therefore, the right-hand side of (26) can be easily rewritten into (20), i.e.,

$$x(t) = \frac{H\pi}{\Omega} \sum_{k=-\infty}^{\infty} X_k \cdot b_k(t).$$

□

Below, we give another expression for reconstruction formula (20) in terms of generating function b of partition $\{b_k,\, k \in \mathbb{Z}\}$.

Corollary 1. *Let function x fulfill the assumptions of Theorem 2. Then, x can be reconstructed from its F-transform components so that*

$$x(t) = \frac{h}{H} \sum_{k=-\infty}^{\infty} X_k \cdot b\left(\frac{t - t_k}{H}\right), \tag{27}$$

where $b \in L_2(\mathbb{R})$ is a generating function of the adjoint (h, H)-uniform partition $\{b_k,\, k \in \mathbb{Z}\}$.

Remark 2. If in (27), we assume that $H = h$ (in other words, $\{a_k,\, k \in \mathbb{Z}\}$ is an h-uniform fuzzy partition of $\mathbb{R}$), then the reconstruction from the F-transform components takes the form

$$x(t) = \sum_{k=-\infty}^{\infty} X_k \cdot b\left(\frac{t - t_k}{h}\right) = \sum_{k=-\infty}^{\infty} X_k \cdot b\left(\frac{t}{h} - k\right), \tag{28}$$

where $b \in L_2(\mathbb{R})$ is the function whose Fourier transform is equal to

$$\widehat{b}(\omega) = \frac{\mathbf{1}_{[-\pi,\pi]}}{\widehat{a}(\omega)}. \tag{29}$$

Reconstruction (28) is similar to the Nyquist-Shannon-Kotelnikov formula (2).

In the next corollary, we extend the range of applicability of Theorem 2 to the h-uniform partition $\{\delta_k,\, k \in \mathbb{Z}\}$ introduced in the Example 1. By this, we obtain the Nyquist-Shannon-Kotelnikov reconstruction in the form of (2).

Corollary 2. *Let the assumptions of Theorem 2 be fulfilled and the Dirac's delta δ and* sinc *be chosen as generating function of an h-uniform partition $\{\delta_k,\, k \in \mathbb{Z}\}$ and the corresponding adjoint h-uniform partition $\{\mathrm{sinc}_k,\, k \in \mathbb{Z}\}$. Then, after respective substitutions the reconstruction formula (28) becomes equivalent with the Nyquist-Shannon-Kotelnikov reconstruction in the form of (2).*

Proof. In the Example 1, we characterize the adjoint h-uniform partition of $\mathbb{R}$ with respect to the h-uniform partition $\{\delta_k,\, k \in \mathbb{Z}\}$. According to (19), this adjoint partition is given by the set of translations $\{\mathrm{sinc}_k,\, k \in \mathbb{Z}\}$ of the generating function sinc. Let us substitute sinc for b in (28) and obtain

$$x(t) = \sum_{k=-\infty}^{\infty} X_k \cdot \mathrm{sinc}(\frac{t - t_k}{h}). \tag{30}$$

By (6),

$$X_k = \frac{\int_{-\infty}^{\infty} \delta_k(s) \cdot x(s)\, ds}{\int_{-\infty}^{\infty} \delta_k(s)\, ds} = x(t_k), \quad k \in \mathbb{Z},$$

so that we can substitute $x(t_k)$ for X_k in (30) and see that the latter becomes equivalent with (2). □

Remark 3. The principal difference between the Nyquist-Shannon-Kotelnikov and the proposed reconstruction is that the former one works as an interpolating technique, while the latter one is able to perform reconstruction even from averaged values of a given function.

5 Conclusion

We discussed the problem of reconstruction from a set of F-transform components. We introduced the adjoint fuzzy partition and the inversion formula and proved that a function can be reconstructed from its F-transform components. Moreover, we showed that if the Dirac's delta δ is chosen as generating function of an h-uniform partition, then the reconstruction from the F-transform components becomes equivalent with the Nyquist-Shannon-Kotelnikov reconstruction.

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