How to Estimate Resilient Modulus for Unbound Aggregate Materials: A Theoretical Explanation of an Empirical Formula

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***Abstract*—To ensure the quality of pavement, it is important to make sure that the resilient moduli – that describe the stiffness of all the pavement layers – exceed a certain threshold. From the mechanical viewpoint, pavement is a non-linear medium. Several empirical formulas have been proposed to describe this non-linearity. In this paper, we describe a theoretical explanation for the most accurate of these empirical formulas.**

I. Formulation of the Problem

Need for estimating resilient modulus. To ensure the quality of a road, it is important to make sure that all the pavement layers have reached a certain stiffness level. To characterize stiffness of unbound pavement materials, transportation engineers use *resilient modulus* M_r.

A material's resilient modulus is actually an estimate of its modulus of elasticity E, i.e., of ratio of stress by strain; the difference from the usual modulus of elasticity if that:

- the usual modulus corresponds to a *slowly* applied load, while
- the resilient characterizes the effect of *rapidly* applied loads – like those experienced by pavements.

A precise definition of the resilient modulus is given, e.g., in [1].

Need to take non-linearity into account. In the usual (*linear*) elastic materials, the modulus does not depend on the stress value. In contrast, pavement materials are usually *non-linear*, in the sense that the resilient stress depends on the stress.

Empirical formulas describing pavement's non-linearity. Several empirical formulas have been proposed to describe this dependence. Experimental comparison [2] shows that the best description is provided by the formula (first proposed in [3])

$$M_r = k_1' \cdot \left(\frac{\theta}{P_a} + 1\right)^{k_2'} \cdot \left(\frac{\tau_{\rm oct}}{P_a} + 1\right)^{k_3'},$$

where P_a is atmospheric pressure, θ is the *bulk stress*, i.e., the trace

$$\theta = \sum_{i=1}^{3} \sigma_{ii}$$

of the stress tensor σ_{ij} (see, e.g., [4]), and

$$\tau_{\rm oct} \stackrel{\rm def}{=} \sqrt{\frac{1}{3} \cdot \sum_{ij} \sigma_{ij}^2 - \frac{1}{3} \cdot \theta^2}$$

is the *octahedral shear stress*.

In terms of the eigenvalues σ_1, σ_2, and σ_3 of the stress tensor,

$$\theta = \sigma_1 + \sigma_2 + \sigma_3$$

and

$$\tau_{\rm oct} = \frac{1}{3} \cdot \sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}.$$

What we do in this paper. In this talk, we provide a theoretical explanation for the above empirical formula.

This explanation uses the general idea that the fundamental physical formulas should not change if we simply changing the measuring unit and/or the starting point for the measurement scale.

Paper outline. First, in Section 2, we briefly explain the general idea, that fundamental physical formulas should not depend on the choice of the starting point or on the choice of the measuring unit.

In Section 3, we use this general idea to describe possible dependence of the resilient modulus M_r on, correspondingly, the bulk stress θ and on the octahedral sheer stress $\tau_{\rm oct}$.

Finally, in Section 4, we apply similar ideas to combine the two formulas for $M_r(\theta)$ and $M_r(\tau_{\rm oct})$ into a single formula $M(\theta, \tau_{\rm oct})$ that describes the dependence of the resilient modulus on both stresses.

Comment. In our derivation, we are not using physical equation, we are only using expert knowledge – which, in this case, is formulated in terms of invariance. From this viewpoint, this paper can be viewed as a particular case of soft computing, techniques for formalizing and utilizing expert knowledge.

II. General Idea: Fundamental Physical Formulas Should Not Depend on the Choice of the Starting Point or of the Measuring Unit

Main idea. Computers process numerical values of different quantities. A numerical value of a quantity depends on the choice of a measuring unit and – in many cases – also on the choice of the starting point.

For example, depending on the choice of a measuring unit, we can describe the height of the same person as 1.7 m or 170 cm. Similarly, we can describe the same moment of time as 2 pm (14.00) if we use El Paso time or 3 pm (15.00) if we use Austin time – the difference is caused by the fact that the starting points for these two times – namely midnight (00.00) in El Paso and midnight (00.00) in Austin – differ by one hour.

The choice of a measuring unit is rather arbitrary. For example, we can measure length in meters or in centimeters or in feet. Similarly, the choice of the starting point is arbitrary: when we analyze a cosmic event, it does not matter the time of what location we use to describe it. It is therefore reasonable to require that the fundamental physical formulas not depend on the choice of a measuring unit and – if appropriate – on the choice of the starting point. We do not expect that, e.g., Newton's laws look differently if we use meters or feet.

Of course, if we change the units in which we measure one of the quantities, then we may need to adjust units of related quantities. For example, if we replace meters with centimeters, then for the formula $v = d/t$ (that describes velocity v as a ratio of distance d and time t) to remain valid we need to replace meters per second with centimeters per second when measuring velocity. However, once the appropriate adjustments are made, we expect the formulas to remain the same.

Not all physical quantities allow both changes. It should be mentioned that while most physical quantities do not have any preferred measuring unit – and thus, selection of a different measuring unit makes perfect physical sense – some quantities have a fixed starting point. For example, while we can choose an arbitrary starting point for time, for distance, 0 distance seems to be a reasonable starting point.

As a result, while the change of a measuring unit makes sense for most physical quantities, the change of a starting point only makes sense for some of them – and a physics-based analysis is needed to decide whether this change makes physical sense.

How to describe the change of a measuring unit in precise terms. If we replace the original measuring unit with a new unit which is a times smaller, then all numerical values of the measured quantity get multiplied by a: $x' = a \cdot x$.

For example, if we replace meters with centimeters – which are $a = 100$ times smaller – then the original height of $x = 1.7$ m becomes $x' = a \cdot x = 100 \cdot 1.7 = 170$ cm.

How to describe the change of the starting point in precise terms. If we replace the original starting point by a new one which is b earlier (or smaller), then to all numerical values of the measured quantity the value b is added: $x' = x + b$.

For example, if we replace El Paso time with Austin time – which is $b = 1$ hour earlier, then the original time of $x = 14.00$ hr becomes $x' = x + b = 14.00 + 1.00 = 15.00$ hr.

In general, we can change both the measuring unit and the starting point. If we first change the measuring unit and the starting point, then:

- first, the original value x first gets multiplied by a, resulting in $x' = a \cdot x$, and
- then the value b is added to the new value x', resulting in $x'' = x' + b = a \cdot x + b$.

Thus, in general, when we change both the measuring unit and the starting point, we get a linear transformation $x \to a \cdot x + b$.

III. How Resilient Modulus Depends on the Bulk Stress (and on the Octahedral Shear Stress)

What we do in this section. Let us first use the above idea to describe how the resilient modulus M_r depends on the bulk stress θ.

Which invariances makes sense in this case. As we have mentioned in the previous section,

- while the change of a measuring unit makes sense for (practically) *all* physical quantities,
- the change of the starting point only makes physical sense for *some* quantities.

Let us therefore analyze whether the change of the starting point makes sense for the resilient modulus M_r and for the bulk stress θ.

For the resilient modulus, there is a clear starting point $M_r = 0$, in which strain does not cause any stress. So, for the resilient modulus, only a change in a measuring unit makes physical sense.

In contrast, for the bulk stress, we can clearly have several choices of the starting point, choices motivated by the fact that in addition to the external stress, there is also an always-present atmospheric pressure. One possibility is to only count the external stress and thus, consider the situation in which we only have atmospheric pressure as corresponding to zero stress. Another possibility is to explicitly take atmospheric pressure into account and take the ideal vacuum no-atmospheric-pressure situation as zero stress. In the first case, we can select atmospheric pressures corresponding to different heights as different starting points.

What does it mean for the resulting formula to be independent: first approximation. For the dependence $M_r(\theta)$, the requirement that this dependence does not change if we change numerical values of θ means the following. For every

$a > 0$ and b, the dependence in the new units $M_r(a \cdot \theta + b)$ has exactly the same form as in the old units – if we also appropriately re-scale M_r. So, we should have

$$M_r(a \cdot \theta) + b = c(a, b) \cdot M_r(\theta) \tag{1}$$

for some value c which, in general, depends on a and b.

What are the functions that satisfy this condition: analysis of the problem. Let us find all the functions $M_r(\theta)$ for which, for some function $c(a, b)$, the equality (1) holds for all x, $a > 0$, and b.

From the physical viewpoint, small changes in θ should lead to small changes in M_r, i.e., in mathematical terms, the dependence $M_r(\theta)$ should be continuous. It is know that every continuous function can be approximated, with any given accuracy, by a differentiable function (e.g., by a polynomial). Thus, without losing generality, we can safely assume that the dependence $M_r(\theta)$ is differentiable.

Thus, the function

$$c(a, b) = \frac{M_r(a \cdot \theta + b)}{M_r(\theta)}$$

is also differentiable, as a ratio of two differentiable functions. For $a = 1$, the formula (1) takes the form

$$M_r(\theta + b) = c(1, b) \cdot M_r(\theta). \tag{2}$$

Differentiating both sides of formula (2) with respect to b and setting $b = 0$, we get

$$M_r'(\theta) = c \cdot M_r(\theta), \tag{3}$$

where $f'(x)$ denote the derivative, and c is the derivative of $c(1, b)$ with respect to b for $b = 0$.

The equation (3) can be rewritten as

$$\frac{dM_r}{d\theta} = c \cdot M_r,$$

i.e., equivalently, as

$$\frac{dM_r}{M_r} = c \cdot d\theta.$$

Integrating both sides, we get $\ln(M_r) = c \cdot \theta + C_0$ for some constant C_0. Thus,

$$M_r = A \cdot \exp(c \cdot \theta), \tag{4}$$

where $A \stackrel{\text{def}}{=} \exp(C_0)$.

For $b = 0$ and $a \neq 0$, the equation (1) takes the form

$$M_r(a \cdot \theta) = c(a, 0) \cdot M_r(\theta).$$

Substituting the expression (4) into this formula, we conclude that

$$A \cdot \exp(c \cdot a \cdot \theta) = c(a, 0) \cdot \exp(c \cdot \theta). \tag{5}$$

When $c \neq 0$, the two sides grow with θ at a different speed, so we should have $c = 0$ and $M_r(\theta) = \text{const}$.

Thus, the only case when the formula $M_r(\theta)$ is fully invariant is when we have a linear material, with $M(\theta) = \text{const}$.

Since we cannot require all the invariances, let us require only some of them. Since we cannot require invariance with respect to *all* possible re-scalings, we should require invariance with respect to *some* family of re-scalings.

If a formula does not change when we apply each transformation, it will also not change if we apply them one after another, i.e., if we consider a composition of transformations. Each shift can be represented as a superposition of many small (infinitesimal) shifts, i.e., shifts of the type $\theta \to \theta + B \cdot dt$ for some B. Similarly, each re-scaling can be represented as a superposition of many small (infinitesimal) re-scalings, i.e., re-scalings of the type $\theta \to (1 + A \cdot dt) \cdot \theta$. Thus, it is sufficient to consider invariance with respect to an infinitesimal transformation, i.e., a linear transformation of the type

$$\theta \to \theta' = (1 + A \cdot dt) \cdot \theta + B \cdot dt.$$

Invariance means that the value $M(\theta')$ has the same form as $M(\theta)$, i.e., that $M(\theta')$ is obtained from $M(\theta)$ by an appropriate (infinitesimal) re-scaling $M_r \to (1 + C \cdot dt) \cdot M_r$. In other words, we require that

$$M_r((1 + A \cdot dt) \cdot \theta + B \cdot dt) = (1 + C \cdot dt) \cdot M_r(\theta), \tag{6}$$

i.e., that

$$M_r(\theta + (A \cdot \theta + B) \cdot dt) = M_r(\theta) + C \cdot M_r(\theta) \cdot dt.$$

Here, by definition of the derivative, $M_r(\theta + q \cdot dt) = M_r(\theta) + M_r'(\theta) \cdot q \cdot dt$. Thus, from (6), we conclude that

$$M_r(\theta) + (A \cdot \theta + B) \cdot M_r'(\theta) \cdot dt = M_r(theta) + C \cdot M_r(\theta) \cdot dt.$$

Subtracting $M_r(\theta)$ from both sides and dividing the resulting equality by dt, we conclude that

$$(A \cdot \theta + B) \cdot M_r'(\theta) = C \cdot M_r(\theta).$$

Since $M_r'(\theta) = \frac{dM_r}{d\theta}$, we can separate the variables by moving all the terms related to M_r to one side and all the terms related to θ to another side. As a result, we get

$$\frac{dM_r}{M_r} = C \cdot \frac{d\theta}{A \cdot \theta + b}.$$

Degenerate cases when $A = 0$ can be approximated, with any given accuracy, by cases when A is small but non-zero. So, without losing generality, we can safely assume that $A \neq 0$. In this case, for $x \stackrel{\text{def}}{=} \theta + k$, where $k \stackrel{\text{def}}{=} \frac{B}{A}$, we have

$$\frac{dM_r}{M_r} = c \cdot \frac{dx}{x},$$

where $c \stackrel{\text{def}}{=} \frac{C}{A}$. Integration leads to $\ln(M_r) = c \cdot \ln(\theta) + C_0$ for some constant C_0, thus $M_r = C_1 \cdot x^c$ for $C_1 \stackrel{\text{def}}{=} \exp(C_0)$, i.e.,

$$M_r(\theta) = C_1 \cdot (\theta + k)^c. \tag{7}$$

Dependence on the bulk stress: conclusion. If we represent $\theta+k$ as $k\cdot\left(\dfrac{\theta}{k}+1\right)$, then we get the desired dependence of M_r on θ:

$$M_r = C_2\cdot\left(\frac{\theta}{k}+1\right)^c, \tag{8}$$

where $C_2 \stackrel{\text{def}}{=} C_1\cdot k^c$.

Dependence on the octahedral sheer stress. Similarly, we can conclude that the dependence $M_r(\tau_{\text{oct}})$ of the resilient modulus M_r on the octahedral sheer stress τ_{oct} has the form

$$M_r = C'_2\cdot\left(\frac{\tau_{\text{oct}}}{k'}+1\right)^{c'}, \tag{9}$$

for some constants C'_2, k', and c'.

IV. How to Combine The Formulas Describing Dependence on Each Quantities into a Formula Describing Joint Dependence

Idea. We have used the invariance ideas to derive formulas $M_r(\theta)$ and $M_r(\tau_{\text{oct}})$ describing dependence of M_r on each of the quantities θ and τ_{oct}. Let us now use the same ideas to combine these two formulas into a single formula describing the dependence on both quantities θ and τ_{oct}.

Based on the previous analysis, for each pair $(\theta,\tau_{\text{oct}})$, we know the value of the modulus M_r:

- the value $M_1 \stackrel{\text{def}}{=} M_r(\theta)$ that we obtain if we ignore the octahedral sheer stress and only take into account the bulk stress; and
- the value $M_2 \stackrel{\text{def}}{=} M_r(\tau_{\text{oct}})$ that we obtain if ignore the bulk stress and only take into account the octahedral sheer stress.

Based on these two values M_1 and M_2, we would like to compute an estimate $M(M_1,M_2)$ for the modulus that would take into account both inputs.

All three values M, M_1, and M_2 represent modulus. Thus, for all three values, only scaling is possible. So, the invariance requirement takes the following form: for every p and q, if we apply the re-scalings $M_1\to p\cdot M_1$ and $M_2\cdot q\cdot M_2$, then the resulting dependence $M(p\cdot M_1, q\cdot M_2)$ has the same form as the original dependence $M(M_1,M_2)$ – after an appropriate re-scaling by some parameter $c(p,q)$ depending on p and q.

So, for every p and every q, there exists a $c(p,q)$ for which, for all M_1 and M_2, we have

$$M(p\cdot M_1, q\cdot M_2) = c(p,q)\cdot M(M_1,M_2). \tag{10}$$

Analysis of the problem. If we re-scale only one of the inputs, e.g., M_1, we get

$$M(p\cdot M_1, M_2) = c_1(p)\cdot M(M_1,M_2), \tag{11}$$

where $c_1(p)\stackrel{\text{def}}{=} c(p,1)$. If we first re-scale by p and then by p', then this is equivalent to one re-scaling by $p\cdot p'$. In the first case, we get

$$M((p\cdot p')\cdot M_1, M_2) = M(p'\cdot(p\cdot M_1), M_2) =$$

$$c_1(p')\cdot M(p\cdot M_1, M_2) = c_1(p')\cdot c_1(p)\cdot M(M_1,M_2). \tag{12}$$

In the second case, we get

$$M((p\cdot p')\cdot M_1, M_2) = c_1(p\cdot p')\cdot M(M_1,M_2). \tag{13}$$

Since the left-hand sides of the equalities (12) and (13) are equal, their right-hand sides must be equal as well. Dividing the resulting equality by $M(M_1,M_2)$, we conclude that

$$c_1(p\cdot p') = c_1(p)\cdot c_1(p'). \tag{14}$$

Differentiating this equality by p' and taking $p'=1$, we conclude that

$$p\cdot c'_1(p) = c_0\cdot c_1(p),$$

where $c_0\stackrel{\text{def}}{=} c'_1(1)$. Thus,

$$\frac{dc_1}{c_1} = c_0\cdot\frac{dp}{p},$$

so integration leads to $\ln(c_1) = c_0\cdot\ln(p)+\text{const}$, and

$$c_1(p) = \text{const}\cdot p^{c_0}. \tag{15}$$

For $M_1 = 1$, the formula (11) takes the form

$$M(p, M_2) = \text{const}\cdot p^{c_0}\cdot M(1,M_2), \tag{16}$$

i.e., renaming the variable,

$$M(M_1,M_2) = \text{const}\cdot M_1^{c_0}\cdot M(1,M_2). \tag{17}$$

Similarly, we have

$$M(M_1,M_2) = \text{const}'\cdot M_2^{c'_0}\cdot M(M_1,1), \tag{18}$$

for some constants const' and c'_0. In particular, for $M_1=1$, the formula (18) takes the form

$$M(1,M_2) = \text{const}'\cdot M_2^{c'_0}\cdot M(1,1). \tag{19}$$

Substituting this expression into the formula (17), we get

$$M(M_1,M_2) = \text{const}\cdot M_1^{c_0}\cdot\text{const}'\cdot M_2^{c'_0}\cdot M(1,1). \tag{30}$$

Substituting expressions (8) and (9) for M_1 and M_2 into this formula, we come up with the following conclusion.

Conclusion. From the invariance requirements, we can conclude that the dependence of M_r on θ and τ_{oct} has the form

$$M(\theta,\tau_{\text{oct}}) = k_1\cdot\left(\frac{\theta}{k}+1\right)^{k_2}\cdot\left(\frac{\tau_{\text{oct}}}{k'}+1\right)^{k_3},$$

where $k_2 = c\cdot c_0$, $k_3 = c'\cdot c'_0$, and

$$k_1 = \text{const}\cdot\text{const}'\cdot M(1,1)\cdot C_2^c\cdot (C'_2)^{c'}.$$

Thus, we indeed get a theoretical explanation for the empirical dependence.

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