Decision Making Under Interval Uncertainty as a Natural Example of a Quandle

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Abstract

In many real-life situations, we need to select an alternative from a set of possible alternatives. In many such situations, we have a well-defined objective function $u(a)$ that describes our preferences. If we know the exact value of $u(a)$ for each alternative a, then we select the alternative with the largest value of $u(a)$. In practice, however, we usually know the consequences of each decision a only with some uncertainty. As a result, for each alternative a, instead of the exact utility value $u(a)$, we only know the interval of possible values $[\underline{u}(a), \overline{u}(a)]$. In this paper, we show that the resulting problem of decision making under interval uncertainty is a natural example of a *quandle*, i.e., of a general class of operations introduced in knot theory.

1 Need for Decision Making under Interval Uncertainty

Need for decision making. In many real-life situations, we need to select an alternative a from the list of possible alternatives – e.g., we want to select a design and/or location of a plant, a financial investment, etc.

In many such situations, we have a well-defined objective function $u(a)$ that describes our preferences. If we know the exact value of $u(a)$ for each alternative a, then we select the alternative with the largest value of $u(a)$.

Decision making under interval uncertainty. In practice, we usually only know the consequences of each decision with some uncertainty. Often, the only information that we have about the corresponding values of $u(a)$ is that it is somewhere between the known bounds $\underline{u}(a)$ and $\overline{u}(a)$, i.e., that $u(a) \in [\underline{u}(a), \overline{u}(a)]$.

How can we make a decision under such interval uncertainty?

To make a decision under interval uncertainty, we need to select a value from the interval. To make a decision under interval uncertainty, we need, in particular, to be able to compare:

- the alternative a for which we only know the interval of possible values of the objective function $u(a)$,
- with alternatives b for which we know the exact utility values $u(b)$.

For some values $u(b)$, the alternative b is better; for others, a is better. Clearly, if a is better than b and $u(b) > u(c)$, then a should be better than c as well. Similarly, if a is worse than b and $u(b) < u(c)$, then a should be worse than c as well. Thus, there should be a threshold value u_0 that separates alternatives b for which a is better from alternatives b' for which a is better.

In other words, when we make decisions, we compare $u(b)$ with this threshold value u_0. This value u_0 thus represents the equivalent utility of the alternative for which we only know the interval $[\underline{u}(a), \overline{u}(a)]$.

We therefore need to be able, given an interval $[\underline{u}(a), \overline{u}(a)]$, to produce an equivalent utility value u_0. In the following text, we will denote this value u_0 by $\overline{u}(a) \rhd \underline{u}(a)$.

Main problem: which operation $\rhd$ should we select?

2 Natural Properties of the Corresponding Operation $\rhd$

In order to answer the above questions, let us analyze what are the natural properties of the operation $a \rhd b$.

Case of a degenerate interval. First, if we know the exact value of $u(a)$, i.e., if the corresponding interval has the form $[x, x]$ for some x, then the corresponding equivalent value is simply equal to x:

$$x \rhd x = x \quad (1).$$

Monotonicity. Another reasonable property is *monotonicity:* if $x < x'$, then $x \rhd y < x' \rhd y$.

Continuity. Small changes in x and y should lead to small changes in the equivalent value $x \rhd y$. In other words, the operation $\rhd$ should be *continuous*.

Case of twin interval uncertainty. In practice, instead of knowing the exact bounds $\underline{u}(a)$ and $\overline{u}(a)$ on $u(a)$, we may only know the bounds on each of these bounds: e.g., we know that $\underline{u}(a) \in [\underline{u}^-(a), \underline{u}^+(a)]$ and $\overline{u}(a) \in [\overline{u}^-(a), \overline{u}^+(a)]$. Such a situation is known as *twin interval uncertainty*; see, e.g., [4, 10].

For example, we may know the lower bound z of the corresponding interval, but we do not know its upper bound: we only know that this upper bound is between y and x. We can analyze this situations in two different ways.

First, we can say that since all we know about the upper bound is that it is between y and x, this upper bound is therefore equivalent to the value $y \triangleright x$. Now, after we have thus reduced the uncertain upper bound to a single number, the original information becomes simply an interval with an exact lower bound z and an exact upper bound $x \triangleright y$. We can now apply the operation $\triangleright$ to estimate the equivalent value of this interval as $(x \triangleright y) \triangleright z$.

There is also an alternative approach. For each possible value v between y and x, we have an interval $[z, v]$ with equivalent value $v \triangleright z$. Due to the natural monotonicity, this equivalent value is the smallest when v is the smallest, i.e., when $v = y$, and it is the largest when v is the largest, i.e., when $v = x$. Thus, possible equivalent values form an interval $[y \triangleright z, x \triangleright z]$. The equivalent value of this interval is therefore $(x \triangleright z) \triangleright (y \triangleright z)$.

It is reasonable to require that these two approaches lead to the same value, i.e., that

$$(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z) \tag{2}$$

Similarly, we can consider situations in which we know the upper bound x of the corresponding interval, but we do not know its lower bound: we only know the lower bound is between y and z. In this case, a similar analysis leads to the requirement that

$$x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z). \tag{3}$$

Bounds. If we know that u is in the interval $[\underline{u}, \overline{u}]$, this means that u is not worse than $\underline{u}$. Thus, we should have $u_0 \geq \underline{u}$.

Similarly, from the fact that $u \leq \overline{u}$, we conclude that $u_0 \leq \overline{u}$. Thus, in general, we should have $x \triangleright y \in [y, x]$.

This is a quandle. Interestingly, the above three natural properties (1)-(3) (plus an appropriately formulated monotonicity) are well known in knot theory: sets with operations satisfying these properties are knows as *quandles*; see, e.g., [3, 9].

Let us use this relation to describe possible operations $\triangleright$ for decision making under interval uncertainty.

3 Main Result

Discussion. In general, the operation $\triangleright$ is monotonically increasing with respect to each of its variables. For differentiable functions, this implies that both partial derivatives are non-negative. Our result, however, requires a stronger condition: that both derivatives are always positive.

We also need to require not only that $x \triangleright y \in [y, x]$, but also that $x \triangleright y \in (y, x)$ for $y < x$, i.e., that the degenerate cases $x \triangleright y = x$ and $x \triangleright y = y$ are excluded.

Under these conditions, we prove the following result.

Definition 1. *We say that a differentiable function* $f(x_1, \ldots, x_m)$ *is* strongly increasing *if all its partial derivatives are positive.*

Proposition 1. *Let $x \rhd y$ be a continuously differentiable strongly increasing function defined for all $x \geq y$ which satisfies (1), satisfies (2) or (3), and for which $x \rhd y \in (y, x)$ when $y < x$. Then,*

$$x \rhd y = f^{-1}(\alpha \cdot f(x) + (1 - \alpha) \cdot f(y)) \tag{4}$$

for some continuous strictly increasing function $f(x)$ and for some $\alpha \in (0, 1)$.

Proof. The proof is, in effect, the same as the proof given in Section 7.2.4 Part C of [1] for a similar result in the case when the operation $x \rhd y$ is defined for all possible pairs of real numbers (x, y).

Discussion. In other words, after an appropriate monotonic re-scaling

$$x \to X = f(x),$$

we get

$$X \rhd Y = \alpha \cdot X + (1 - \alpha) \cdot Y.$$

This way of making decisions under interval uncertainty is well known: it has been originally proposed by the Nobelist Leo Hurwicz and is thus known as *Hurwicz's optimism-pessimism criterion*; see, e.g., [5, 8, 7, 11]. This criterion makes intuitive sense: it means that to make a decision, we consider, with different weights, the best-case outcome X and the worst-case outcome Y.

Our result provides a new justification for Hurwicz's criterion, with one important exception: by requiring that $x \rhd y \in (y, x)$ when $y < x$, we exclude the following two extreme cases:

- the super-optimistic case $\alpha = 1$, when the decision maker only takes into account the best-case situation; and
- the super-pessimistic case $\alpha = 0$, when the decision maker only takes into account the worst-case situation.

Open questions. What if we only require that $x \rhd y \in [y, x]$? What if we only require monotonicity – and allow zero values of the derivatives? What if we only require continuity instead of differentiability?

4 What If We Also Allow Improper Intervals?

Need for improper intervals. In interval uncertainty, in addition to usual intervals $[a, b]$ with $a \leq b$, it is sometimes useful to consider *improper* intervals $[a, b]$, with $a > b$.

The need for such improper intervals comes, e.g., from the following situation. Let us consider the case when a decision maker is participating in two different situations. In the first situation, the decision maker gains some amount u, about which we only know that $u \in [\underline{u}, \overline{u}]$ for some bounds $\underline{u}$ and $\overline{u}$. In the second situation, the decision maker gains some amount v about which we only know that $v \in [\underline{v}, \overline{v}]$ for some bounds $\underline{v}$ and $\overline{v}$. As a result of both situations, the possible values of the amount $u + v$ gained by the decision maker form an interval $[\underline{u} + \underline{v}, \overline{v} + \overline{v}]$.

Suppose now that after different decision makers participates in the first situations and gain some amount $u \in [\underline{u}, \overline{u}]$, we want to compensate them so that at the end, each of them will gain the exact overall amount $\underline{u} + \overline{u}$ (which is equal to double the average gain). How can we describe the corresponding compensation v?

We do not know beforehand the value of this compensation v, it depends on how much the decision maker will gain in the first situation. Depending on the main, the corresponding compensation can range:

- from the smallest possible value $v = \underline{u}$ – which corresponds to the case when the decision maker's gain in the first situation was the largest $u = \overline{u}$
- to the largest possible value $v = \overline{u}$ – which corresponds to the case when the decision maker's gain in the first situation was the smallest $u = \underline{u}$.

So, at first glance, it may seem that the possible values of the compensation v can also be described by the interval $[\underline{u}, \overline{u}]$. However, this will lead us to the conclusion that the possible values of overall gain $u + v$ form the interval $[\underline{u} + \underline{u}, \overline{u} + \overline{u}]$ – and we want to describe the compensation in which the overall gain is always equal to $\underline{u} + \overline{u}$.

To avoid this erroneous conclusion, it makes sense to say that the possible values of the compensation amount v form an *improper* interval $[\overline{u}, \underline{u}]$; see, e.g., [6, 12]. In this case, if we apply the above formula to describe possible values of $u \in [\underline{u}, \overline{u}]$ and $v \in [\overline{u}, \underline{u}]$, then for the overall gain $u + v$, we get the interval

$$[\underline{u} + \overline{u}, \overline{u} + \underline{u}],$$

i.e., we conclude – correctly this time – that the overall compensation is always equal to $\underline{u} + \overline{u}$.

Resulting question. It is reasonable to extent the question of selecting an appropriate value u_0 to such improper intervals as well. In this case, the operation $x \rhd y$ is defined for all possible pairs of real numbers (x, y).

Results. It turns out that if we allow improper intervals, then we can relax some of the restrictions that we placed on the operation $\rhd$ in Proposition 1 – but for that, we need to require that *both* conditions (2) and (3) are satisfied:

Proposition 2. [2] *If a function $x \rhd y$ is continuous, strictly increasing w.r.t. each of its variables, and satisfies (2) and (3), then*

$$x \rhd y = f^{-1}(\alpha \cdot f(x) + (1 - \alpha) \cdot f(y))$$

for some continuous strictly increasing function $f(x)$ and for some $\alpha \in (0, 1)$.

Mathematical comment. In Proposition 2, we assume that *both* requirements (2) and (3) are satisfied. What if only one of them is satisfied? It turns out that a similar result is still true, if we require either differentiability or invertibility of $\triangleright$:

Proposition 3. ([1], Theorem 7.2.5) *If a function $x \triangleright y$ is differentiable, strictly increasing w.r.t. each of its variables, and satisfies (2) or (3), then*

$$x \triangleright y = f^{-1}(\alpha \cdot f(x) + (1 - \alpha) \cdot f(y))$$

for some differentiable strictly increasing function $f(x)$ and for some $\alpha \in (0, 1)$.

Proposition 4. [13] *If a function $x \triangleright y$ is continuous, strictly increasing w.r.t. each of its variables, satisfies (2) or (3), and satisfies the additional property that for every x and y, there exist z' and z'' for which $x \triangleright z' = z'' \triangleright x = y$, then*

$$x \triangleright y = f^{-1}(\alpha \cdot f(x) + (1 - \alpha) \cdot f(y))$$

for some continuous strictly increasing function $f(x)$ and for some $\alpha \in (0, 1)$.

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