From Universal Approximation to Universal Representation: It All Depends on the Continuum Hypothesis

Mahdokht Michelle Afravi and Vladik Kreinovich
Department of Computer Science, University of Texas at El Paso
El Paso, TX 79968, USA
mafravi@miners.utep.edu, vladik@utep.edu

In many practical situations, there is a correlation between two quantities x and y, and the only information that we have to describe this correlation are expert statements formulated in terms of imprecise ("fuzzy") words from natural language, such as "small". For example, an expert can say that if x is small, then y is big, and vice versa. Fuzzy logic is a technique that translates this knowledge into precise mathematical terms. In this technique, each fuzzy term A is described by a function $A(x)$ assigning, to each possible value x of the corresponding quantity, a degree $A(x)$ to which this value has the appropriate property (e.g., is small). Once we have rules $A_i(x) \Rightarrow B_i(y)$, the degree $d(x, y)$ to which each pair (x, y) is possible can be described as the degree to which either the first rule is satisfied (i.e., $A_1(x)$ and $B_1(y)$) or the second rule is satisfied, etc. One possible way to interpret "and" is to use product, and "or" is sum. Then, the desired degree takes the form $d(x, y) = \sum\limits_{i=1}^{n} A_i(x) \cdot B_i(y)$.

It is known that this expression has a universal approximation property: for every $\varepsilon > 0$, every continuous function on a box can be ε-approximated by such sums. A natural question is: when can we get an *exact* representation of every function? Of course, we cannot do it with finitely many terms, we may need infinitely many, but it is reasonable to require that the sum is always well defined, i.e., that for every pair (x, y), only finitely many terms are different from 0. Surprisingly, the corresponding universal representation property cannot be proved or disproved in basic set theory: it is equivalent to Continuum Hypothesis, which is itself independent from the axioms of set theory.

References

[1] R. O. Davies, "Representation of functions of two variables as sums of rectangular functions I", *Fundamenta Mathematicae*, 1974, Vol. 85, No. 2, pp. 177–183.