Which Confidence Set Is the Most Robust?

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Formulation of the problem. Once we have a probability distribution, then for each confidence level α, we can form a *confidence set* S, i.e., a set S for which $P(S) = \alpha$. There are many such sets, which one should we choose?

It is reasonable to select S to be connected. For each connected confidence set, we can define the degree of robustness by finding out how much the probability changes if we change the set slightly around some point x on the border ∂S of the set S. If we add or subtract a small piece of volume ΔV, then the change in probability ΔP can be found from the definition of the probability density function (pdf) $\rho(x) \stackrel{\text{def}}{=} \lim\limits_{\Delta V \to 0} \dfrac{\Delta P}{\Delta V}$, as $\Delta P = \rho(x) \cdot \Delta V + o(\Delta V)$. Thus, for fixed (and small) ΔV, this change is proportional to $\rho(x)$.

The value of $\rho(x)$ is, in general, different for different points $x \in \partial S$. As a measure of robustness of the given set S, it is reasonable to select the worst-case value $r(S) \stackrel{\text{def}}{=} \max\limits_{x \in \partial S} \rho(x)$. How can we select the confidence set with the smallest possible value of $r(S)$?

Solution. We show that the smallest possible value is attained when $\rho(x)$ is constant on ∂S. This means that for usual distributions like Gaussian, the corresponding confidence set is $S = \{x : \rho(x) \geq \rho_0\}$ for some ρ_0. For Gaussian distributions, such an ellipsoid is indeed usually selected as a confidence set.

Idea of the proof. If the function $\rho(x)$ is not constant, then we can expand S slightly in the vicinity of points x_0 for which $\rho(x_0) \approx \max\limits_{x} \rho(x)$ – thus making the border value of $\rho(x)$ smaller, and at the same time shrink S slightly near all other points $x \in \partial S$, to compensate for the increase in $P(S)$ caused by expanding. After this shrinking, the border values of $\rho(x)$ at the corresponding points x will increase – but if we shrink a little bit, these values will still be smaller than $\max\limits_{x} \rho(x)$. This way, we will get a new confidence set for which $\max\limits_{x} \rho(x)$ is slightly smaller.

Thus, for the set S at which $r(S)$ is the smallest, the values of pdf cannot be non-constant on the border. So, for the optimal (most robust) confidence set, we indeed have $\rho(x) = \text{const}$ for all $x \in \partial S$.