It Is Possible to Determine Exact Fuzzy Values Based on an Ordering of Interval-Valued Fuzzy Degrees

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***Abstract*—In the usual $[0,1]$-based fuzzy logic, the actual numerical value of a fuzzy degree can be different depending on a scale, what is important – and scale-independent – is the order between different values. To make a description of fuzziness more adequate, it is reasonable to consider interval-valued degrees instead of numerical ones. Here also, what is most important is the order between the degrees. If we have only order between the intervals, can we, based on this order, reconstruct the original numerical values – i.e., the degenerate intervals? In this paper, we show that such a reconstruction is indeed possible, moreover, that it is possible under three different definitions of order between numerical values.**

I. Formulation of the Problem

Need for probabilities and need for fuzzy degrees. To describe how frequently different events occur, a natural idea is to use probabilities – i.e., in effect, frequencies with which this event has occurred. For example, if under certain weather conditions, in the past, rain happens in 30% of the cases, we say that under these conditions, the probability of rain is 30%.

In general, if out of n cases, the event of interest happened in m of them, we say that the probability of the event is equal to m/n. Strictly speaking, this frequency is only an approximation to the actual probability; the larger the sample size n, the more accurate this approximation. So, if we want a more accurate estimate for the probability, we need to increase n, i.e., to consider a larger sample.

In addition to this objective probability, there is also subjective uncertainty: experts are not 100% sure about their statements. For example, an expert may say that the probability of rain under certain conditions is *small*, without providing a precise numerical description of what this word means.

To process this subjective uncertainty, we need to describe in computer-understandable terns, i.e., in terms of numbers. This description is one of the main objectives of fuzzy logic; see, e.g., [1], [4], [5]. To describe an imprecise ("fuzzy") word like "small" in precise terms, we can, e.g., for each possible value x of the corresponding quantity (in the above example, for each possible probability value) to estimate, on a given scale, how certain this expert is that x is small. If the expert marked his or her degree of certainty as 7 on a scale from 0 to 10, we can then say that the expert's degree of certainty is 7/10. In general, if an expert marked m on a scale from 0 to n, we can use the value m/n.

An important difference between probabilities and fuzzy degrees. In both cases of probabilistic and fuzzy uncertainty, we have the same formula m/n for estimating the corresponding degree. However, there is a big difference between these values.

Probabilities are objective. If two people use the same data, they will get the same probability value.

In contrast, expert opinions are subjective. Based on the same evidence, and based on the same understanding of what is more probable and what is less probable, some experts will be more "optimistic" and mostly use values close to 10 on a scale from 0 to 10, while other experts may be more "pessimistic" and mostly use values close to 0 on the same scale. In this case, the values m/n corresponding to one expert can be transformed into values m'/n corresponding to another expert by an appropriate non-linear re-scaling.

As a result, in fuzzy logic, the actual numerical value of a fuzzy degree can be different depending on a scale. What is important – and scale-independent – is the *order* between different values.

Need for interval-valued fuzzy logic. Fuzzy logic deals with situations in which an expert uses imprecise words like "small" to describe his or her opinion. The expert uses imprecise words because he or she is unable to come up with an exact estimate.

On the other hand, the traditional $[0,1]$-based fuzzy logic requires the same expert to come up with an exact value of a scale from 0 to 10 that describes this expert's degree of certainty. Of course, in practice, the expert is often unable to do it. To be more precise, the expect may be able to confidently say that his.her degree of certainty is 7 and not 6 and not 8, but if we try to get a more accurate description by taking a scale form, say, 1 to 100, it is doubtful that the expert will be able to mark his/her degree of confidence as 71 and not

70 or 72. At best, the expert would be able to mark a whole *interval* of possible values – e.g., from 65 to 75 – as describing his/her degree of certainty. This corresponds to the interval $[0.65, 0.75]$ of possible degree.

Such interval-valued fuzzy techniques have indeed been proposed. They are indeed more adequate in describing expert's uncertainty, and they have led to many practical applications; see, e.g., [2], [3].

Interval values generalize the usual fuzzy logic: each degree $a \in [0, 1]$ from the original fuzzy logic can also be viewed as a "degenerate" interval $[a, a]$ in the interval-valued fuzzy scheme – but, of course, in interval-valued approach, we have additional degrees $[a, b]$ with $a < b$.

Formulation of the problem. In the interval-valued case, also re-scalings are possible. As a result, what is most important is the numerical values, but rather the *order* between the degrees.

If we have only order between the intervals, can we, based on this order, reconstruct the original numerical values – i.e., the degenerate intervals?

What we do in this paper. In this paper, we show that such a reconstruction is indeed possible, moreover, that it is possible under three different definitions of order between numerical values.

II. First Ordering: Lattice (Component-Wise) Order

Component-wise order between intervals: a brief reminder. If for some statement, the expert's degree of confidence is represented by an interval $[a, b]$, and then we increase the lower bound, to make the interval $[a', b]$ with $a' > a$, we thus increase our degree of confidence in this statement. Similarly, if we increase b to $b' > b$, we thus increase our degree of confidence. From this viewpoint, it makes sense to say that the interval $[a', b']$ represents a larger)or same) degree of confidence than the interval $[a, b]$ if $a' \geq a$ and $b' > b$:

$$[a, b] \leq [a', b'] \Leftrightarrow (a \leq a' \,\&\, b \leq b').$$

Formulation of the problem in precise terms. Suppose that on the set of all subintervals $[a, b]$ of the interval $[0, 1]$, we have the above ordering.

Based on this ordering, can we uniquely determine degenerate intervals, i.e., intervals of the type $[a, a]$? In this section, we will answer that this is indeed possible. This determination will be done step by step.

First step: it is possible to define the interval $[0, 0]$ based only on the order. Indeed, $[0, 0]$ is the only interval which is smaller (in the sense of the above relation $\leq$) than any other interval.

In precise terms, the interval $[0, 0]$ is the only interval I that satisfies the property

$$\forall J\, (I \leq J),$$

where variables I and J go over intervals.

Second step: it is possible to defined intervals of the type $[0, a]$ based only on the order. Our claim is that an interval I is of type $[0, a]$ if and only if the set of all intervals between $[0, 0]$ and I is linearly ordered, i.e., if and only if

$$\forall J\, \forall J'\, (([0, 0] \leq J \leq I \,\&\, [0, 0] \leq J' \leq I) \Rightarrow (J \leq J' \vee J' \leq J)).$$

Indeed, if $I = [0, a]$, then for each interval $J = [b, c]$, the condition

$$[0, 0] \leq J = [b, c] \leq [0, a]$$

implies that $0 \leq b \leq 0$ and thus, that $b = 0$. So, all such intermediate intervals J and J' have the form $[0, c]$ for some real value c.

Of course, all such intervals are linearly ordered. Indeed, for $J = [0, c]$ and $J' = [0, c']$, either $c \leq c'$ or $c' \leq c$.

- In the first case, we have $J \leq J'$.
- In the second case, we have $J' \leq J$.

Let us show that, vice versa, if the interval I has the form $[a, b]$ with $a \neq 0$ (i.e., $a > 0$), then there exist J and J' between $[0, 0]$ and I for which $J \not\leq J'$ and $J' \not\leq J$. Indeed, it is sufficient to take $J = [a, 0]$ and $J' = [0, b]$.

Final step: it is possible to define degenerate intervals based only on the order. We already know how to define a degenerate interval $[0, 0]$.

Our claim is that I is a degenerate interval $[a, a]$, with $a > 0$ if and only if I is *not* of the type $[0, a]$ and there exists an interval I' of the type $[0, a]$ for which

- $I' \leq I$,
- the set of all intervals J between I' and I is linearly ordered, and
- for no larger interval $I'' \geq I$, $I'' \neq I$, the set of all intervals J between I' and I'' is linearly ordered.

Indeed, if $I = [a, a]$ for some $a > 0$, then we can take $I' = [0, a]$ for this same a. Then all intervals J and J' between I' and I have the form $[b, a]$ for some b and the same a and are, thus, linearly ordered.

On the other hand, if $I'' = [a'', b'']$ is larger than $I = [a, a]$, this means that either $a'' > a$ – in which case $b'' \geq a'' > a$ and thus $b'' > a$ – or $b'' > a$. Then, both $J = I$ and $J' = [0, b'']$ are between I' and I'', but $J \not\leq J'$ and $J' \not\leq J$.

Vice versa, let us assume that I is a non-degenerate interval $[a_0, b_0]$ for some $a_0 < b_0$. the fact that this is not an interval of type $[0, a]$ means that $a_0 > 0$. In this case, linear ordering for all intervals J and J' between $I' = [0, a]$ and $I = [a_0, b_0]$ is only possible if $b_0 = a$. Indeed, if $b_0 = a$, then we do get the linear ordering, but if $b_0 > a$, then the intervals $J = [0, b_0]$ and $J' = [a_0, a]$ are between I' and I, but $J \not\leq J'$ and $J' \leq J$.

So, if there is a linear ordering of all intervals between I' and I, then $b_0 = a$, and the interval I has the form $[a_0, a]$, with $a_0 < a$. However, now we can take a larger interval $I'' = [a, a] \geq I$, and still be able to conclude that all intervals between I' and I'' are linearly ordered – which contradicts to our requirement that no such larger interval is possible.

Thus, the above condition indeed uniquely determines degenerate intervals.

III. Second Ordering: Necessarily Larger

Description of the "necessarily larger" ordering. The fact that for a statement S, instead of a single fuzzy value we have an interval $[a, b]$ of possible fuzzy values can be interpreted as saying that the actual (unknown) expert's degree of confidence in this statement can be any value between a and b.

Similarly, for another statement S', the corresponding interval $[a', b']$ means that the actual (unknown) expert's degree of confidence in the statement S' can be any value between a' and b'.

A reasonable idea is to ask when we can be absolutely certain that our degree of belief in S is smaller than or equal to the degree of belief in S'. Since we only know that the intervals that contain the actual values, the only way to be absolutely certain is to make sure that each value from the interval $[a, b]$ is smaller than or equal to any value from the interval $[a', b']$.

This means, in particular, that $b \le a'$, Vice versa, if $b \le a'$, then any value from the interval $[a, b]$ is smaller than or equal to b and is, thus, smaller than or equal to a'. In its turn, a' is smaller than or equal to any value from the interval $[a', b']$. Thus, indeed, if $b \le a'$, then any value from the interval $[a, b]$ is smaller than or equal to any value from the interval $[a', b']$, So, the "necessarily larger" relation takes the following form:

$$[a, b] \le [a', b'] \Leftrightarrow b \le a'.$$

Formulation of the problem in precise terms. Suppose that on the set of all subintervals $[a, b]$ of the interval $[0, 1]$, we have the above ordering.

Based on this ordering, can we uniquely determine degenerate intervals, i.e., intervals of the type $[a, a]$? In this section, we will answer that this is indeed possible. This determination will be also done step by step.

First step: it is possible to describe interval inclusion based only on the order. Let us first show that the notion of interval inclusion $[a, b] \subseteq [a', b']$ can be described based only on the above-defined order. Namely, we will show that

$$I \subseteq I' \Leftrightarrow$$
$$\forall I'' \, ((I' \le I'' \Rightarrow I \le I'') \,\&\, (I'' \le I' \Rightarrow I'' \le I)).$$

Indeed, let us assume that $I \subseteq I'$. For every interval I'', the relation $I' \le I''$ means that every element from the interval I' is smaller than or equal to every element from the interval I''. Since $I \subseteq I'$, every element of I is also an element of I' and is, thus, smaller than or equal to every element of I''. This means that $I \le I''$.

Similarly, for every interval I'', the relation $I'' \le I'$ means that every element from the interval I'' is smaller than or equal to every element from the interval I'. Since $I \subseteq I'$, every element of I is also an element of I' and is, thus, larger than or equal to every element of I''. This means that $I'' \le I$.

Vice versa, let us assume that for $I = [a, b]$ and $I' = [a', b']$, we have

$$\forall I'' \, ((I' \le I'' \Rightarrow I \le I'') \,\&\, (I'' \le I' \Rightarrow I'' \le I)).$$

In particular, for $I'' = [b', b']$, we have $I' \le I''$ and thus, we have $I \le I''$, i.e., $[a, b] \le [b', b']$. According to our description of the "necessarily larger" relation, this means that $b \le b'$.

Similarly, for $I'' = [a', a']$, we have $I'' \le I'$ and thus, we have $I'' \le I$, i.e., $[a', a'] \le [a, b]$. According to our description of the "necessarily larger" relation, this means that $a' \le a$.

So, $a' \le a \le b \le b'$, which means exactly that $[a, b] \subseteq [a', b']$.

Final step: it is possible to define degenerate intervals based only on the order. We already know how to define inclusion in terms of the order.

A degenerate interval I can then be defined as the one that does not have any subinterval different from itself:

$$\forall J (J \subseteq I \Rightarrow J = I).$$

IV. Third Ordering: Possibly Larger

Description of the "possibly larger" ordering. Another reasonable idea is to ask when it is *possible* that our degree of belief in S is smaller than or equal to the degree of belief in S'. Since we only know that the intervals that contain the actual values, this means that there exists a value from the interval $[a, b]$ which is smaller than or equal to some value from the interval $[a', b']$.

If $v \le v'$ for some v and v' for which $a \le v \le b$ and $a' \le v' \le b'$, then from $a \le v \le v' \le b'$, we conclude that $a \le b'$.

Vice versa, if $a \le b'$, then we have values $a \in [a, b]$ and $b' \in [a', b']$ for which $a \le b'$ and thus, $[a, b]$ is possibly smaller than $[a', b']$. So, the "possibly larger" relation takes the following form:

$$[a, b] \le [a', b'] \Leftrightarrow a \le b'.$$

Formulation of the problem in precise terms. Suppose that on the set of all subintervals $[a, b]$ of the interval $[0, 1]$, we have the above ordering.

Based on this ordering, can we uniquely determine degenerate intervals, i.e., intervals of the type $[a, a]$? In this section, we will answer that this is indeed possible. This determination – similarly to the two previous cases – will be done step by step.

First step: it is possible to describe interval inclusion based only on the order. Let us first show that the notion of interval inclusion $[a, b] \subseteq [a', b']$ can be described based only on the above-defined order. Namely, we will show that

$$I \subseteq I' \Leftrightarrow$$
$$\forall I'' \, ((I \le I'' \Rightarrow I' \le I'') \,\&\, (I'' \le I \Rightarrow I'' \le I')).$$

Indeed, let us assume that $I \subseteq I'$. For every interval I'', the relation $I \le I''$ means that some element from the interval

I is smaller than or equal to some element from the interval I''. Since $I \subseteq I'$, every element of I is also an element of I'. Thus, some element of I' is smaller than or equal than some element of I''. This means that $I' \leq I''$.

Similarly, for every interval I'', the relation $I'' \leq I$ means that some element from the interval I'' is smaller than or equal to some element from the interval I. Since $I \subseteq I'$, every element of I is also an element of I'. Thus, some element of I'' is smaller than or equal than some element of I'. This means that $I'' \leq I'$.

Vice versa, let us assume that for $I = [a, b]$ and $I' = [a', b']$, we have

$$\forall I''\,((I \leq I'' \Rightarrow I' \leq I'')\,\&\,(I'' \leq I \Rightarrow I'' \leq I')).$$

In particular, for $I'' = [a, a]$, we have $I \leq I''$ and thus, we have $I' \leq I''$, i.e., $[a', b'] \leq [a, a]$. According to our description of the "possibly larger" relation, this means that $a' \leq a$.

Similarly, for $I'' = [b, b]$, we have $I'' \leq I$ and thus, we have $I'' \leq I'$, i.e., $[b, b] \leq [a', b']$. According to our description of the "possibly larger" relation, this means that $b \leq b'$.

So, $a' \leq a \leq b \leq b'$, which means exactly that $[a, b] \subseteq [a', b']$.

Final step: it is possible to define degenerate intervals based only on the order. We already know how to define inclusion in terms of the order.

Then, similarly to the case of "necessarily larger" relation, we can define a degenerate interval I as the one that does not have any subinterval different from itself:

$$\forall J(J \subseteq I \Rightarrow J = I).$$

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