Isn't Every Sufficiently Complex Logic Multi-Valued Already: Lindenmabum-Tarski Algebra and Fuzzy logic Are Both Particular Cases of the Same Idea

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***Abstract*—Usually, fuzzy logic (and multi-valued logics in general) are viewed as drastically different from the usual 2-valued logic. In this paper, we show that while on the surface, there indeed seems to be a major difference, a more detailed analysis shows that even in the theories based on the 2-valued logic, there naturally appear constructions which are, in effect, multi-valued, constructions which are very close to fuzzy logic.**

I. Formulation of the Problem

We typically view fuzzy logic as drastically different from the traditional 2-valued logic. Our knowledge of the world is rarely absolutely perfect. As a result, when we make decisions, then, in addition to the well-established facts, we have to rely on the human expertise, i.e., on expert statements about which the experts themselves are not 100% confident.

If we had a perfect knowledge, then, for each possible statement, we would know for sure whether this statement is true or false. Since our knowledge is not perfect, for many statements, we are not 100% sure whether they are true or false. To describe and process such statements, Zadeh proposed special *fuzzy logic* techniques, in which, in addition to "true" and "false", we have intermediate degrees of certainty; see, e.g., [3], [6], [7].

In a nutshell, the main idea behind fuzzy logic is to go:

- from the traditional 2-valued logic, in which every statement is either true or false,
- to a multi-valued logic, in which we have more options to describe our opinion about he truth of different statements.

From this viewpoint, the traditional 2-valued logic and the fuzzy logic are drastically different: namely, they correspond to a different number of possible truth values.

If we take Gödel's theorem into account, the difference becomes less drastic. At first glance, the difference does seem drastic. However, let us recall that the above description of the traditional 2-valued logic is based on the idealized case when for every statement S, we know whether this statement is true or false. This is possible in simple situations, but, as the famous Gödel's theorem shows, such an idealized situation is not possible for sufficiently complex theories; see, e.g., [2], [5]. Namely, Gödel proved that already for arithmetic – i.e., for statements obtained from basic equality and inequality statements about polynomial expressions by adding propositional connectives $\&$, $\vee$, $\neg$, and quantifiers over natural numbers – it is not possible to have a theory T in which for every statement S, either this statement or its negation are derived from this theory (i.e., either $T \models S$ or $T \models \neg S$).

Thus, there exist statements S for which $T \not\models S$ and $T \not\models \neg S$. So:

- while, legally speaking, the corresponding logic is 2-valued,
- in reality, such a statement S is neither true nor false – and thus, we have more than 3 possible truth values.

At first glance, it may seem that here, we have a 3-valued logic, with possible truth values "true", "false", and "unknown", but in reality, we may have more, since:

- while different "true" statements are all provably equivalent to each other, and
- all "false" statements are provably equivalent to each other,
- different "unknown" statements are not necessarily provably equivalent to each other.

To get a more adequate description of this situation, it is reasonable to consider the equivalence relation $\models (A \Leftrightarrow B)$ between statements A and B.

Equivalence classes with respect to this relation can be viewed as the actual truth values of the corresponding theory. The set of all such equivalence classes is known as the *Lindenbaum-Tarski algebra*; see, e.g., [2], [5].

But what does this have to do with fuzzy logic? Lindenbaum-Tarski algebra shows that any sufficiently complex logic is, in effect, multi-valued. However, this multi-valuedness is different from the multi-valuedness of fuzzy logic.

What we do in this paper. In this paper, we show that there is another aspect of multi-valuedness of the traditional logic, an aspect of which the usual fuzzy logic is a particular case. Thus, we show that the gap between the traditional 2-valued logic and the fuzzy logic is even less drastic.

II. Our Idea

Need to consider several theories. In the above text, we considered the case when we have a single theory T.

Gödel's theorem states that for every given theory T that includes formal arithmetic, there is a statement S that can neither be proven nor disproven in this theory. Since this statement S can neither be proven not disproven based on the axioms of theory T, a natural idea is to consider additional reasonable axioms that we can add to T.

This is what happened, e.g., in geometry, when it turned out that the V-th postulate – that for every line ℓ in a plane and for every point P outside this line, there exists only one line ℓ' which passes through P and is parallel to ℓ. Since neither this statement nor any its negation can be derived from all other (more intuitive) axioms of geometry, a natural solution is to explicitly add this statement as a new axiom. (If we add its negation, we get Lobachevsky geometry – historically the first non-Euclidean geometry; see, e.g., [1].)

Similarly, in set theory, it turns out that the Axiom of Choice and Continuum Hypothesis cannot be derived or rejected based on the other (more intuitive) axioms of set theory; thus, they (or their negations) have to be explicitly added to the original theory; see, e.g., [4].

The new – extended – theory covers more statements that the original theory T.

- However, the same Gódel's theory still applies.
- So for the new theory, there are still statements that can neither be deduced nor rejected based on this new theory.
- Thus, we need to add one more axiom, etc.

As a result:

- instead of a *single* theory,
- it makes sense to consider a *family* of theories $\{T_\alpha\}_\alpha$.

In the above description, we end up with a family which is *linearly ordered* in the sese that for every two theories T_α and T_β, either $T_\alpha \models T_\beta$ or $T_\beta \models T_\alpha$. However, it is possible that on some stage, different groups of researchers select two different axioms – e.g., a statement and its negation. In this case, we will have two theories which are not derivable from each other – and thus a family of theories which is not linearly ordered.

How is all this applicable to expert knowledge? From the logical viewpoint, processing expert knowledge can also be viewed as a particular case of the above scheme: axioms are the basic logical axioms + all the expert statements statements that we believe to be true.

- We can select only the statements in which experts are 100% sure, and we get one possible theory.
- We can add statements for which expert's degree of confidence exceeds a certain threshold – and get a different theory, with a larger set of statements.
- Depending on our selection of the threshold, we thus get different theories T_α.

So, in fact, we also have a family of theories $\{T_\alpha\}_\alpha$, where different theories T_α correspond to different levels of the certainty threshold.

Once we have a family of theories, how can we describe the truth of a statement? If we have a single theory T, then for every statement S, we have three possible options:

- either $T \models S$, i.e., the statement S is true in the theory T,
- or $T \models \neg S$, i.e., the statement S is false in the theory T,
- or $T \not\models S$ and $T \not\models \neg S$, i.e., the statement S is undecidable in this theory.

Since, as we have mentioned earlier, a more realistic description of our knowledge means that we have to consider a family of theories $\{T_\alpha\}_\alpha$, it is reasonable to collect this information based on all the theories T_α.

Thus, to describe whether a statement S is true or not, instead of a single yes-no value (as in the case of a single theory), we should consider the values corresponding to all the theories T_α, i.e., equivalently, we should consider the whole set

$$\deg(S) \stackrel{\text{def}}{=} \{\alpha : T_\alpha \models S\}.$$

This set is our degree of belief that the statement S is true – i.e., in effect, the truth value of the statement S.

Logical operations on the new truth values. If a theory T_α implies both S and S', then it implies their conjunction $S \,\&\, S'$ as well. Thus, the truth value of the conjunction includes the intersection of truth value sets corresponding to S and S':

$$\deg(S \,\&\, S') \supseteq \deg(S) \cap \deg(S').$$

Similarly, if a theory T_α implies either S or S', then it also implies the disjunction $S \vee S'$. Thus, the truth value of the disjunction includes the union of truth value sets corresponding to S and S':

$$\deg(S \vee S') \supseteq \deg(S) \cup \deg(S').$$

What happens in the simplest case, when the theories are linearly ordered? If the theories T_α are linearly ordered, then, once $T_\alpha \models S$ and $T_\beta \models T_\alpha$, we also have $T_\beta \models S$. Thus, with every T_α, the truth value $\deg(S) = \{\alpha : T_\alpha \models S\}$ includes, with each index α, the indices of all the stronger theories – i.e., all the theories T_β for which $T_\beta \models T_\alpha$.

In particular, in situations when we have a finite family of theories, each degree if equal to $D_{\alpha_0} \stackrel{\text{def}}{=} \{\alpha : T_\alpha \models T_{\alpha_0}\}$ for some α_0. In terms of the corresponding linear order

$$\alpha \le \beta \Leftrightarrow T_\alpha \models T_\beta,$$

this degree takes the form $D_{\alpha_0} = \{\alpha : \alpha \le \alpha_0\}$. We can thus view α_0 as the degree of truth of the statement S:

$$\text{Deg}(S) \stackrel{\text{def}}{=} \alpha_0.$$

In case of expert knowledge, this means that we consider the smallest degree of confidence d for which we can derive the statement S if we allow all the expert's statements whose degree of confidence is at least d.

- If we can derive S by using only statements in which the experts are absolutely sure, then we are very confident in this statement S.
- On the other hand, if, in order to derive the statement S, we need to also consider expert's statement in which the experts are only somewhat confident, then, of course, our degree of confidence in S is much smaller.

These sets D_α are also linearly ordered: one can easily show that

$$D_\alpha \subseteq D_\beta \Leftrightarrow \alpha \le \beta.$$

In this case:

- the intersection of sets D_α and D_β simply means that we consider the set $D_{\min(\alpha,\beta)}$, and
- the union of sets D_α and D_β simply means that we consider the set $D_{\max(\alpha,\beta)}$.

Thus, the above statements about conjunction and disjunction take the form

$$\text{Deg}(S \,\&\, S') \ge \min(\text{Deg}(S), \text{Deg}(S'));$$

$$\text{Deg}(S \vee S') \ge \max(\text{Deg}(S), \text{Deg}(S')).$$

This is very similar to the usual fuzzy logic. The above formulas are very similar to the formulas of the fuzzy logic corresponding to the most widely used "and"- and "or"-operations: $\min$ and $\max$. (The only difference is that we get $\ge$ instead of the equality.)

Thus, fuzzy logic can be indeed naturally obtained in the classical 2-valued environment: namely, it can be interpreted as a particular case of the same general idea as the Lindenbaum-Tarski algebra.

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