

Practical Need for Equation-Type Interval Equations and for Their Extended-Zero Solutions

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Abstract. One of the main problems in interval computations is solving systems of equations under interval uncertainty. Usually, interval computation packages consider united, tolerance, and control solutions. In this paper, we explain the practical need for equation-type solutions, when we look for solutions for which both sides are equal. In situations when such a solution is not possible, we provide a justification for extended-zero solutions, in which we ignore intervals of the type $[-a, a]$.

1 Practical Need for Solving Interval Systems of Equations: What Is Known

Need for data processing. In many practical situations, we are interested in the values of quantities $y_1, \dots, y_m$ which are difficult – or even impossible – to measure directly. For example, we can be interested in a distance to a faraway star or in tomorrow’s temperature at a certain location.

Since we cannot measure these quantities directly, to estimate these quantities we must:

- find easier-to-measure quantities $x_1, \dots, x_n$ which are related to y_i by known formulas $y_i = f_i(x_1, \dots, x_n)$,
- measure these quantities x_j , and
- use the results $\tilde{x}_j$ of measuring the quantities x_j to compute the estimates for y_i :

$$\tilde{y}_i = f(\tilde{x}_1, \dots, \tilde{x}_n).$$

Computation of these estimates is called *indirect measurement* or *data processing*.

Need for data processing under uncertainty. Measurements are never 100% accurate. Hence, the measurement result $\tilde{x}_j$ is, in general, different from the

actual (unknown) value x_j of the corresponding quantity; in other words, the measurement errors $\Delta x_j \stackrel{\text{def}}{=} \tilde{x}_j - x_j$ are, in general, different from 0.

Because of the non-zero measurement errors, the estimates $\tilde{y}_i$ are, in general, different from the desired values y_i . It is therefore desirable to know how accurate are the resulting estimates.

Need for interval uncertainty and interval computations. The manufacturer of the measuring instrument usually provides us with an upper bound Δ_j on the measurement error: $|\Delta x_j| \leq \Delta_j$; see, e.g., [7]. If no such upper bound is known, i.e., if the reading of the instrument can be as far away from the actual value as possible, then this is not a measuring instrument, this is a wild-guess-generator.

Sometimes, we also know the probabilities of different values Δx_j within this interval; see, e.g., [7, 13]. However, in many practical situations, the upper bound is the only information that we have [7]. In this case, after we know the result $\tilde{x}_j$ of measuring x_j , the only information that we have about the actual (unknown) value x_j is that this value belongs to the interval $[\underline{x}_j, \bar{x}_j]$, where $\underline{x}_j \stackrel{\text{def}}{=} \tilde{x}_j - \Delta_j$ and $\bar{x}_j \stackrel{\text{def}}{=} \tilde{x}_j + \Delta_j$.

In this case, the only thing that we can say about each value $y_i = f_i(x_1, \dots, x_n)$ is that this value belongs to the range

$$\{f_i(x_1, \dots, x_n) : x_1 \in [\underline{x}_1, \bar{x}_1], \dots, x_n \in [\underline{x}_n, \bar{x}_n]\}.$$

Computation of this range is one of the main problems of *interval computations*; see, e.g., [3, 6].

Sometimes, we do not know the exact dependence. The above text described an ideal case, when we know the exact dependence $y_i = f_i(x_1, \dots, x_n)$ between the desired quantities y_i and the easier-to-measure quantities x_j . In practice, often, we do not know the exact dependence. Instead, we know that the dependence belongs to a finite-parametric *family* of dependencies, i.e., that

$$y_i = f_i(x_1, \dots, x_n, a_1, \dots, a_k)$$

for some parameters $a_1, \dots, a_k$.

For example, we may know that y_i is a linear function of the quantities x_j , i.e., that $y_i = c_i + \sum_{j=1}^n c_{ij} \cdot x_j$ for some coefficients c_i and c_{ij} .

The presence of these parameters complicates the corresponding data processing problem. Depending on what we know about the parameters, we have different situations.

Simplest situation, when we know the exact values of all the parameters. The simplest situation is when we know the exact values of these parameters. In this case, the dependence of y_i on x_j is known, and we have the same problem of computing the range as before.

Specific case: control solution. Sometimes, not only we *know* the values a_ℓ of these parameters, but we can also *control* these values, by setting them to any

values within certain intervals $[\underline{a}_\ell, \bar{a}_\ell]$. By setting the appropriate values of the parameters, we can change the values y_i . This possibility naturally leads to the following problem:

- we would like the values y_i to be within some given ranges $[\underline{y}_i, \bar{y}_i]$; for example, we would like the temperature to be within a comfort zone;
- in this case, it is desirable to find the range of possible values of x_j for which, by applying appropriate controls $a_i \in [\underline{a}_\ell, \bar{a}_\ell]$, we can place the values y_i within these intervals.

In the degenerate case, when all the intervals for y_i and a_ℓ are just points, this means solving the system of equations $y = f(x, a)$, where we denoted $y \stackrel{\text{def}}{=} (y_1, \dots, y_m)$, $x \stackrel{\text{def}}{=} (x_1, \dots, x_n)$, and $a \stackrel{\text{def}}{=} (a_1, \dots, a_k)$. From this viewpoint, the above problem can be viewed as an interval generalization of the problem of solving a system of equations, or, informally, as a problem of solving the corresponding interval system of equations.

The set X of all appropriate values $x = (x_1, \dots, x_n)$ can be formally described as

$$X = \{x : \text{for some } a_\ell \in [\underline{a}_\ell, \bar{a}_\ell], f_i(x_1, \dots, x_n, a_1, \dots, a_k) \in [\underline{y}_i, \bar{y}_i] \text{ for all } i\}.$$

This set is known as the *control solution* to the corresponding interval system of equations [3, 12].

Situation when we need to find the parameters from the data. Sometimes, we know that the values a_i are the same for all the cases, but we do not know these values. These values must then be determined based on measurements: we measure x_j and y_i several times, and we find the values of the parameters a_ℓ that match all the measurement results.

Let us number all membership cycles by values $c = 1, \dots, C$. After each cycle of measurements, we conclude that:

- the actual (unknown) value of $x_j^{(c)}$ is in the interval $[\underline{x}_j^{(c)}, \bar{x}_j^{(c)}]$ and
- the actual value of $y_i^{(c)}$ is in the interval $[\underline{y}_i^{(c)}, \bar{y}_i^{(c)}]$.

We want to find the set A of all the values a for which $y^{(c)} = f(x^{(c)}, a)$ for some $x^{(c)}$ and $y^{(c)}$:

$$A = \{a : \forall c \exists x_j^{(c)} \in [\underline{x}_j^{(c)}, \bar{x}_j^{(c)}] \exists y_i^{(c)} \in [\underline{y}_i^{(c)}, \bar{y}_i^{(c)}] (f(x^{(c)}, a) = y^{(c)})\}.$$

This set A is known as the *united solution* to the interval system of equations [3, 12].

What can we do once we have found the range of possible values of a . Once we have found the set A of possible values of a , we can first find the range of possible values of y_i based on the measurement results, i.e., find the range

$$\{f_i(x_1, \dots, x_n, a) : x_j \in [\underline{x}_j, \bar{x}_j] \text{ and } a \in A\}.$$

This is a particular case of the main problem of interval computations.

If we want to make sure that each value y_i lies within the given bounds $[\underline{y}_i, \bar{y}_i]$, then we must find the set X of possible values of x for which $f_i(x, a)$ is within these bounds for all possible values $a \in A$, i.e., the set

$$X = \{x : \forall a \in A \forall i (f_i(x, a) \in [\underline{y}_i, \bar{y}_i])\}.$$

This set is known as the *tolerance solution* to the interval system of equations [3, 12].

Sometimes, we know that the values a may change. In the previous text, we consider the situations when the values a_ℓ are either fixed forever, or can be changed by us. In practice, these values may change in an unpredictable way – e.g., if these parameters represent some physical processes that influence y_i ’s. We therefore do not know the exact values of these parameters, but what we do know is some a priori bounds on these values.

We may know bounds $[\underline{a}_\ell, \bar{a}_\ell]$ on each parameter, in which case the set A of all possible combinations $a = (a_1, \dots, a_k)$ is simply a box:

$$A = [\underline{a}_1, \bar{a}_1] \times \dots \times [\underline{a}_k, \bar{a}_k].$$

We may also have more general sets A – e.g., ellipsoids.

In this case, we can still solve the same two problems whose solutions we described above; namely:

- we can solve the main problem of interval computations – the problem of computing the range – and find the set Y of possible values of y ;
- we can also solve the corresponding tolerance problem and find the set of values x that guarantee that each y_i is within the desired interval.

Is this all there is? There are also more complex problems (see, e.g., [12]), but, in a nutshell, most practical problems are either range estimation, or finding control, united, or tolerance solution. These are the problems solved by most interval computation packages [3, 6].

Is there anything else? In this paper, we show that there is an important class of practical problems that does not fit into one of the above categories. To solve these practical problems, we need to use a different notion of a solution to interval systems of equations.

2 Remaining Problem of How to Find the Set A Naturally Leads to Equality-Type Solutions to Interval System of Equations

Finding the set A : formulation of the problem. In the previous text, we assumed that when the values of the parameter a can change, we know the set A of possible values of the corresponding parameter vector. But how do we find this set?

What information we can use to find the set A . All the information about the real world comes from measurements – either directly from measurements, or by processing measurement results. The only relation between the parameters a and measurable quantities is the formula $y = f(x, a)$. Thus, to find the set A of possible values of a , we need to use measurements of x and y .

We can measure both x and y many times. As a result, we get:

- the set X of possible values of the vector x and
- the set Y of possible values of the vector y .

Both sets can be boxes, or they can be more general sets.

Based on these two sets X and Y , we need to find the set A .

In this problem, it is reasonable to assume that x and a are *independent* in some reasonable sense. Let us formulate this requirement in precise terms.

Independence: towards a formal definition. The notion of independence is well known in the probabilistic case, where it means that probability of getting a value $x \in X$ does not depend on the value $a \in A$: $P(x|a) = P(x|a')$ for all $a, a' \in A$. An interesting corollary of this definition is that, in spite of being formulated in a way that is asymmetric with respect to x and a , this definition is actually symmetric: one can prove that a is independent of x if and only if x is independent of a .

In the interval case, we do not know the probabilities, we only know which pairs (x, a) are possible and which are not. In other words, we have a set $S \subseteq X \times A$ of possible pairs (x, a) . It is natural to say that the values x and a are independent if the set of possible values of x does not depend on a . Thus, we arrive at the following definition.

Definition 1. Let $S \subseteq X \times A$ be a set.

- We say that a pair (x, a) is possible if $(x, a) \in S$.
- Let $x \in X$ and $a \in A$. We say that a value x is possible under a if $(x, a) \in S$. The set of possible-under- a values will be denoted by S_a .
- We say that the variables x and a are independent if $S_a = S_{a'}$ for all a, a' from the set A .

Proposition 1. Variable x and a are independent if and only if S is a Cartesian product, i.e., $S = s_x \times s_a$ for some $s_x \subseteq X$ and $s_a \subseteq A$.

Proof. If $S = s_x \times s_a$, then $S_a = s_x$ for each a and thus, $S_a = S_{a'}$ for all $a, a' \in A$.

Vice versa, let us assume that x and a are independent. Let us denote the common set $S_a = S_{a'}$ by s_x . Let us denote by s_a , the set of all possible values $a \in A$, i.e., the set of all $a \in A$ for which $(x, a) \in S$ for some $x \in X$. Let us prove that in this case, $S = s_x \times s_a$.

Indeed, if $(x, a) \in S$, then, by definition of the set s_x , have $x \in s_x = S_a$, and, by definition of the set s_a , we have $a \in s_a$. Thus, by the definition of the Cartesian product $B \times C$ as the set of all pairs (b, c) of all pairs of elements $b \in B$ and $c \in C$, we have $(x, a) \in s_x \times s_a$.

Vice versa, let $(x, a) \in s_x \times s_a$, i.e., let $x \in s_x$ and $a \in s_a$. By definition of the set s_x , we have $S_a = s_x$, thus $x \in S_a$. By definition of the set S_a , this means that $(x, a) \in S$. The proposition is proven.

As a corollary, we can conclude that the independence relation is symmetric – similarly to the probabilistic case.

Corollary. *Variables x and a are independent if and only if a and x are independent.*

Proof: indeed, both case are equivalent to the condition that the set S is a Cartesian product.

What can we now conclude about the dependence between A , X , and Y . Since we assumed that a and x are independence, we can conclude that the set of possible values of the pair (x, a) is the Cartesian product $X \times A$. For each such pair, the value of y is equal to $y = f(x, a)$. Thus, the set Y is equal to the range of $f(x, a)$ when $x \in X$ and $a \in A$.

The resulting new type of solutions to interval systems of equations. So, we look for sets A for which

$$Y = f(X, A) \stackrel{\text{def}}{=} \{f(x, a) : x \in X \text{ and } a \in A\}.$$

We will call the set A satisfying this property an *equality-type solution* to the interval system of equations.

Such solutions for the interval system of equations $y = f(x, a)$, in which we want the interval versions Y and $f(X, A)$ of both sides of the equation to be exactly equal, are known as *formal solutions*; see, e.g., [1, 2, 5, 10–12].

3 What If the Interval System of Equations Does Not Have an Equality-Type Solution: A Justification for Enhanced-Zero Solutions

But what if an equality-type solution is impossible: analysis of the problem. The description in the previous section seem to make sense, but sometimes, the corresponding problem has no solutions. For example, in the simplest case when $m = n = k = 1$ and $f(x, a) = x + a$, if we have $Y = [-1, 1]$ and $X = [-2, 2]$, then clearly the corresponding equation $Y = X + A$ does not have a solution: no matter what set A we take the width of the resulting interval $X + A$ is always larger than or equal to the width $w(X) = 4$ of the interval X and thus, cannot be equal to $w(Y) = 2$. What shall we do in this case? How can we then find the desired set A ?

Of course, this would not happen if we had the *actual* ranges X and Y , but in reality, we only have estimates for these ranges. So, the fact that we cannot find A means something is wrong with these estimates.

How are ranges X and Y estimated in the first place? To find out what can be wrong, let us recall how the ranges can be obtained from the experiments.

For example, in the 1-D case, we perform several measurements of the quantity x_1 in different situations. Based on the corresponding measurement results $x_1^{(c)}$, we conclude that the interval of possible values must include the set $[\underline{x}_1^{\approx}, \bar{x}_1^{\approx}]$, where $\underline{x}_1^{\approx} \stackrel{\text{def}}{=} \min_c x_1^{(c)}$ and $\bar{x}_1^{\approx} \stackrel{\text{def}}{=} \max_c x_1^{(c)}$. Of course, we can also have some values outside this interval – e.g., for a uniform distribution on an interval $[0, 1]$, the interval formed by the smallest and the largest of the C random numbers is slightly narrower than $[0, 1]$; the fewer measurement we take, the narrower this interval.

So, to estimate the actual range, we *inflate* the interval $[\underline{x}_1^{\approx}, \bar{x}_1^{\approx}]$. In these terms, the fact that we have a mismatch between X and Y means that one of these intervals was not inflated enough.

The values x correspond to easier-to-measure quantities, for which we can make a large number of measurements and thus, even without inflation, get pretty accurate estimates of the actual range X . On the other hand, the values y are difficult to measure; for these values, we do not have as many measurement results and thus, there is a need for inflation.

From this viewpoint, we can safely assume that the range for X is reasonably accurate, but the range of Y needs inflation.

So how do we find A ? In view of the above analysis, if there is no set A for which $Y = f(X, A)$, the proper solution is to inflate each components of the set Y so that the system becomes solvable.

To make this idea precise, let us formalize what is an inflation.

What is an inflation: analysis of the problem. We want to define a mapping I that transforms each non-degenerate interval $\mathbf{x} = [\underline{x}, \bar{x}]$ into a wider interval

$$I(\mathbf{x}) \supset \mathbf{x}.$$

What are the natural properties of this transformation? The numerical value x of the corresponding quantity depends on the choice of the measuring unit, on the choice of the starting point, and – sometimes – on the choice of direction.

- For example, we can measure temperature t_C in Celsius, but we can also use a different measuring unit and a different starting point and get temperatures in Fahrenheit $t_F = 1.8 \cdot t_C + 32$.
- We can use the usual convention and consider the usual signs of the electric charge, but we could also use the opposite signs – then an electron would be a positive electric charge.

It is reasonable to require that the result of the inflation transformation does not change if we simply change the measuring units or change the starting point or change the sign:

- Changing the starting point leads to a new interval $[\underline{x}, \bar{x}] + x_0 = [\underline{x} + x_0, \bar{x} + x_0]$ for some x_0 .
- Changing the measuring unit leads to $\lambda \cdot [\underline{x}, \bar{x}] = [\lambda \cdot \underline{x}, \lambda \cdot \bar{x}]$ for some $\lambda > 0$.
- Changing the sign leads to $-[\underline{x}, \bar{x}] = [-\bar{x}, -\underline{x}]$.

Thus, we arrive at the following definition.

Definition 2. By an inflation, we mean a mapping that maps each non-degenerate interval $\mathbf{x} = [x, \bar{x}]$ into a wider interval $I(\mathbf{x})$ so that:

- for every x_0 , we have $I(\mathbf{x} + x_0) = I(\mathbf{x}) + x_0$;
- for every $\lambda > 0$, we have $I(\lambda \cdot \mathbf{x}) = \lambda \cdot I(\mathbf{x})$; and
- we have $I(-\mathbf{x}) = -I(\mathbf{x})$.

Proposition 2. Every inflation operation has the form

$$[\tilde{x} - \Delta, \tilde{x} + \Delta] \rightarrow [\tilde{x} - \alpha \cdot \Delta, \tilde{x} + \alpha \cdot \Delta]$$

for some $\alpha > 1$.

Comment. A similar result was proven in [4].

Proof. It is easy to see that the above operation satisfies all the properties of an inflation. Let us prove that, vice versa, every inflation has this form.

Indeed, for intervals $\mathbf{x}$ of type $[-\Delta, \Delta]$, we have $-\mathbf{x} = \mathbf{x}$, thus $I(\mathbf{x}) = I(-\mathbf{x})$. On the other hand, due to the third property of an inflation, we should have $I(-\mathbf{x}) = -I(\mathbf{x})$. Thus, for the interval $[\underline{v}, \bar{v}] \stackrel{\text{def}}{=} I(\mathbf{x})$, we should have $-\underline{v}, \bar{v}] = [-\bar{v}, -\underline{v}] = [\underline{v}, \bar{v}]$ and thus, $\underline{v} = -\bar{v}$. So, we have $I([-\Delta, \Delta]) = [-\Delta'(\Delta), \Delta'(\Delta)]$ for some Δ' . Since we should have $[-\Delta, \Delta] \subset I([-\Delta, \Delta])$, we must have

$$\Delta'(\Delta) > \Delta.$$

Let us denote $\Delta'(1)$ by α . Then, $\alpha > 1$ and $I([-1, 1]) = [-\alpha, \alpha]$. By applying the second property of the inflation, with $\lambda = \Delta$, we can then conclude that $I([-\Delta, \Delta]) = [-\alpha \cdot \Delta, \alpha \cdot \Delta]$. By applying the first property of the inflation operation, with $x_0 = \tilde{x}$, we get the desired equality

$$I([\tilde{x} - \Delta, \tilde{x} + \Delta]) = [\tilde{x} - \alpha \cdot \Delta, \tilde{x} + \alpha \cdot \Delta].$$

The proposition is proven.

So how do we find A ? We want to make sure that $f(X, A)$ is equal to the result of a proper inflation of Y .

How can we tell that an interval Y' is the result of a proper inflation of Y ? One can check that this is equivalent to the fact that the difference $Y' - Y$ is a symmetric interval containing 0; such intervals are known as *extended zeros* [8, 9].

Thus, if we cannot find the set A for which $Y = f(X, A)$, we should look for the set A for which the difference $f(X, A) - Y$ is an extended zero.

Historical comment. This idea was first described in [8, 9]; in this paper, we provide a new theoretical justification of this idea.

Multi-D case. What if we have several variables, i.e., $m > 1$? In this case, we may have different inflations for different components Y_i of the set Y , so we should look for the set A for which, for all i , the corresponding difference $f_i(X, A) - Y_i$ is an extended zero.

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