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Fuzzy Analogues of Sets and Functions Can Be Uniquely Determined from the Corresponding Ordered Category: A Theorem

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Abstract: In modern mathematics, many concepts and ideas are described in terms of category theory. From this viewpoint, it is desirable to analyze what can be determined if, instead of the basic category of sets, we consider a similar category of fuzzy sets. In this paper, we describe a natural fuzzy analog of the category of sets and functions, and we show that, in this category, fuzzy relations (a natural fuzzy analogue of functions) can be determined in category terms – of course, modulo 1-1 mapping of the corresponding universe of discourse and 1-1 re-scaling of fuzzy degrees.

Keywords: fuzzy set; ordered category; category of fuzzy sets

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1. Introduction

How to describe possible changes: the notion of categories. The world and all its objects change with time, they change both by themselves and when we perform some actions. For example, if we have a liquid, we can warm it up or cool it down, we can freeze it or boil it, we can mix it with some other liquids. For a company, we can changes it structure by making it more hierarchical or, vice versa, less hierarchical; we can change the employees salaries so that they become more dependent on their performance, we can fire under-performing employees and/or hire new folks.

To be able to describe each specific change, it is desirable to have a general framework for describing all possible transformations that lead to such changes. For the case of the liquid, such transformations include warming up (actually, several different warmings corresponding to different amount of heat), cooling, freezing, boiling, etc.

In addition to applying a single transformation, we can also apply first a transformation f, and then a transformation g. As a result, we get a *composition* $g \circ f$ of two transformations f and g. If we apply first a transformation f, then a transformation g, and after that a transformation h, then the same resulting change can be described in two different ways: as $h \circ (g \circ f)$ and as $(h \circ g) \circ f$. Thus, for the corresponding composition operation $\circ$, we always have associativity $h \circ (g \circ f) = (h \circ g) \circ f$.

In addition to transformations that actually change the object a, we can always consider a "transformation" that keeps the object intact. Such a transformation is usually denoted by id_a. If we add this "transformation" as one of the composition steps, it will not change the result, i.e., we have $f \circ \mathrm{id}_a = \mathrm{id}_a \circ f$.

In addition to transformation that simply change the object, we can also consider the transformations that transform the original object into something else. For example, boiling transforms water into steam, cooking transforms eggs into an omelette, etc. If we take the possibility of several objects into account, then the above scheme becomes what is a known as a *category*; see, e.g., [1]. In the category theory, transformations are known as *morphisms*.

In precise terms, a *category* is a tuple $(\text{Ob}, \text{Mor}, :, \text{id}, \circ)$, where:

- Ob is the set whose elements are called *objects*,
- Mor is a set whose elements are called *morphisms*,
- $:$ Mor $\to$ Ob $\times$ Ob is a mapping that assigns, to each morphism $f \in$ Mor a pair of objects $(a, b) \in$ Ob $\times$ Ob; this is denoted by $f : a \to b$; the object a is called f's *domain*, and b is called f's *range*;
- id is a mapping that assigns, to each object $a \in$ Ob, a morphism $\text{id}_a : a \to a$; and
- $\circ$ is a mapping that assigns, to each pair of morphisms $f : a \to b$ and $g : b \to c$ for which the range of f is equal to the domain of g, a new morphism $g \circ f : a \to c$ so that for every $f : a \to b$, we have $\text{id}_b \circ f = f \circ \text{id}_a = f$.

Category theory is one of the main tools of modern mathematics. Because of its universal character, category theory plays an important role in modern mathematics [1]. Many new mathematical concepts are defined in category terms, and many original concepts are re-formulated in category terms – such a reformulation in very general terms often enables mathematicians to generalize their ideas and results to a more general context.

Different areas of mathematics can be described in terms of different categories:

- Set theory is naturally described in terms of a category Set in which objects are sets and morphisms are functions.
- Topology is described in terms of a category Top in which objects are topological spaces and morphisms are continuous mappings.
- Linear algebra is naturally described in terms of a category Lin, in which objects are linear spaces, and morphisms are linear mappings, etc.

Many mathematical concepts can be reformulated in terms of an appropriate category; see, e.g., [3–5,8–14] and references therein.

What happens in the fuzzy case? If we allow fuzzy sets (see, e.g., see, e.g., [2,7,16,17,19]), what is a natural analog of the category Set? In the category Set, morphisms from a to b are functions. In the crisp case, for each function $f : a \to b$ and for each element $x \in a$, we have a unique value of $y = f(x) \in b$.

Fuzzy means that for each $x \in a$, instead of a single value $y = f(x) \in b$, we may have different possible values $y \in b$, with different degrees of confidence. In general, we can have all possible values $y \in b$. For each $x \in a$ and for each $y \in b$, we have a degree $R_f(x, y) \in [0, 1]$ to which y is a possible value of $f(x)$. Thus, a natural fuzzy analog of a function is a fuzzy relation.

Composition $g \circ f$ of fuzzy relations $f : a \to b$ and $g : b \to c$ can be defined in the usual way. Namely, we want to know, for each pair of elements $x \in a$ and $c \in z$, to what extent there exists a $y \in b$ for which f brings us from a to b and g brings us from y to z. If we interpret "and" as min and there exists (an infinite "or") as max, then the above description translates into the following formula:

$$R_{g \circ f}(x, z) = \max_y \min(R_f(x, y), R_g(y, z)). \tag{1}$$

Since we have fuzzy relations, there is no need to explicitly describe the domain of each morphism: if for some $x \notin a$, the value $f(x)$ is not defined, this simply means that for this x, we have $R_f(x, y) = 0$ for all $y \in b$. Similarly, there is no need to describe the range,

Thus, without losing generality, we can assume that we have only one object – the universal set U, and that the relation $R_f(x, y)$ is defined for all $x \in U$ and $y \in U$. Morphisms are then fuzzy relations, with the usual composition relation (1).

Need for an ordered category. In the crisp case, every property is either true or false.

As we gain more information, we may get more confident in our knowledge. For example, we may start with the situation in which, for a given x, several different values $f(x)$ are possible, but after acquiring new information, we are becoming more and more confident that there is only one possible value y_0 of $f(x)$. This means that for the remaining value y_0, the degree of possibility $R_f(x, y_0)$ remains the same, but for all $y \neq y_0$, the corresponding degree $R_f(x, y)$ decreases. To capture this phenomenon, it is reasonable to supplement the category structure with the corresponding component-wise ordering between fuzzy relations (morphisms): $f \leq f'$ if and only if $R_f(x, y) \leq R_{f'}(x, y)$ for all x and y.

Formulation of the problem. What can be defined based on this category-theory formulation? Can we uniquely determine the elements of the Universe of discourse U and the corresponding relations based on the categorical information?

2. Results

Towards a precise formulation of the problem. It is easy to see that if we have a 1-1 mapping $\pi : U \to U$ of the Universe of discourse U onto itself (i.e., a bijection), then the corresponding transformation $R(x, y) \to R(\pi(x), \pi(y))$ is an *automorphism* of the corresponding category in the sense that it preserves the identity, composition, and order.

Similarly, if we have a 1-1 monotonic mapping $H : [0, 1] \to [0, 1]$, then the transformation $R(x, y) \to H(R(x, y))$ is also such an automorphism. Indeed, since we only consider order between degrees, monotonic transformation of degrees should not change anything.

It turns out that modulo this simple equivalence, we can uniquely determine all the elements $x \in U$ and all the relations $R(x, y)$ from the ordered category, i.e., in precise terms, that every automorphism is a composition of the automorphisms of the above two types. The proof of this result will be based on an explicit description of elements of U and relations $R_f(x, y)$ in category terms.

Let us describe the problem in precise terms.

Definition 1. *By an* ordered category, *we mean a category in which for every two objects a and b, there is a partial order $\leq$ on the set* $\mathrm{Mor}(a, b)$ *of all morphisms from a to b.*

Definition 2. *Let U be a set; we will call it the* Universe of discourse. *By a U-*fuzzy ordered category, *we mean an ordered category in which:*

- *the only object is the set U,*
- *morphisms are fuzzy relations, i.e., mappings $R : U \times U \to [0, 1]$,*
- *the morphism* id *is defined as the mapping for which* $\mathrm{id}(x, x) = 1$ *and* $\mathrm{id}(x, y) = 0$ *for $x \neq y$,*
- *the composition of morphisms is defined by the formula*

$$(g \circ f)(x, z) = \max_y \min(f(x, y), g(y, z)),$$

  and
- *the order between the morphisms is the component-wise order: $f \leq g$ means that $f(x, y) \leq g(x, y)$ for all x and y.*

The U-fuzzy ordered category will be denoted by F_U.

Comment. One can easily see that this is indeed a category, i.e., that the composition of morphisms is associative, and the composition of any morphism f with the identity morphism id is equal to f: $f \circ \mathrm{id} = \mathrm{id} \circ f = f$.

Definition 3. *An* automorphism *of an ordered category is a pair consisting of bijections* $F : \mathrm{Ob} \to \mathrm{Ob}$ *and* $G : \mathrm{Mor} \to \mathrm{Mor}$ *for which:*

- *for all f, a, and b, we have $f : a \to b$ if and only if $G(f) : F(a) \to F(b)$;*
- *for all f and g, we have $G(f \circ g) = G(f) \circ G(g)$,*
- *for all a, we have $G(\mathrm{id}_a) = \mathrm{id}_{F(a)}$, and*

- *for all f and g, we have $f \leq g$ if and only if $G(f) \leq G(g)$.*

Comment. This definition is a natural generalization of the standard definition of automorphism of categories (see, e.g., [6,15,18] to ordered categories.

Proposition. *Let $\pi : U \to U$ be a bijection of U, and let $H : [0,1] \to [0,1]$ be an increasing bijection of the interval $[0,1]$. Then, the mapping $G_{\pi,H}$ that maps each morphism $f(x,y)$ into a morphism $(G_{\pi,H}(f))(x,y) = H(f(\pi(x),\pi(y)))$ is an automorphism of the category F_U.*

Our main result is that these are the only automorphisms of the category F_U.

Theorem. For every set U, every automorphism of the ordered category F_U has the form $G_{\pi,H}$ for some bijection $\pi : U \to U$ and for some monotonic bijection $H : [0,1] \to [0,1]$.

Comment. This may not be very clear from the formulation of the result, but the proof will show that we can determine elements of the set U and values of the mappings $f(x,y)$ in category terms, i.e., we can indeed define fuzzy relations – a natural fuzzy analogue of functions – in category terms.

3. Proofs

3.1. Proof of the Proposition

This proposition is easy to prove: a permutation π does not change anything, and the increasing bijection does not change the order.

3.2. Proof of the Theorem

1°. First, we can describe the morphism f_0 for which $f_0(x,y) = 0$ for all x and y in ordered-category terms, as the only morphism f for which $f \leq g$ for all morphisms g.

Indeed, clearly $f_0 \leq g$ for all g. Vice versa, if $f \leq g$ for all g, then, in particular, $f \leq f_0$, i.e., $f(x,y) \leq f_0(x,y) = 0$ for all x and y, and since $f(x,y) \in [0,1]$, this means that indeed $f(x,y) = 0$ for all x and y.

2°. Let us first characterize all the morphisms $f \neq f_0$ for which the set $\{g : g \leq f\}$ is linearly ordered. Since an automorphism preserves order, every automorphism maps such morphisms into morphisms with the same property.

Specifically, we will prove that a morphism has this property if and only if we have $f(x,y) > 0$ only for one pair (x,y), and we have $f(x',y') = 0$ for all other pairs (x',y').

Indeed, one can easily check that for such morphisms f, the only morphisms $g \leq f$ are the morphisms which also have $g(x',y') = 0$ for all pairs $(x',y') \neq (x,y)$. Such morphisms g are uniquely described by the corresponding value $g(x,y)$. For every two such morphisms g and g', depending on whether $g(x,y) \leq g'(x,y)$ or $g'(x,y) \leq g(x,y)$, we have $g \leq g'$ or $g' \leq g$, i.e., the set $\{g : g \leq f\}$ is indeed linearly ordered.

Vice versa, let us prove that if a morphism has this property, then it has $f(x,y) > 0$ only for one pairs (x,y). Indeed, if we have $f(x,y) > 0$ and $f(x',y') > 0$ for two different pairs $(x,y) \neq (x',y')$, then we would be able to construct two different morphisms $g \leq f$ and $g' \leq f$ for which $g \not\leq g'$ and $g' \not\leq g$. Namely, we take:

- $g(x,y) = f(x,y) > 0$ and $g(x'',y'') = 0$ for all pairs $(x'',y'') \neq (x,y)$, and
- $g'(x,y) = f(x',y') > 0$ and $g(x'',y'') = 0$ for all pairs $(x'',y'') \neq (x',y')$.

This contradicts our assumption that the set $\{g : g \leq f\}$ is linearly ordered.

3°. Let us now describe, in ordered-category terms, morphisms f for which $f(x,x) > 0$ for some $a \in U$ and $f(x',y')$ for all other pairs $(x',y') \neq (x,x)$.

Indeed, out of all morphisms described in Part 2 of this proof, such morphisms can be determined by the additional condition that $f \circ f = f$. This condition is clearly satisfied for such morphisms, while for morphisms for which $f(x,y) > 0$ for some $b \neq a$, the composition $f \circ f$ is, as one can see, identically 0 and thus, different from f.

4°. One can see that two morphisms f and f' of the type described in Part 3 are connected by the relation $\leq$ (i.e., $f \leq f'$ or $f' \leq f$) if and only if they correspond to the same element $a \in U$.

Thus, we can describe elements of the set U in ordered-category terms: as equivalent classes of morphisms of the type described in Part 3 with respect to the relation $(f \leq f') \vee (f' \leq f)$.

Hence, if we have an automorphism, elements are mapped into elements in a 1-1 way, i.e., indeed we have a bijection of the Universe of discourse.

5°. Let us now show that the degrees from the interval $[0,1]$ can also be described – modulo increasing bijections of this interval – in ordered-category terms.

5.1°. Indeed, for each element $a \in U$, different degrees $v \in [0,1]$ can be associated with different morphisms f described in Part 3 of this proof, i.e., morphisms for which:

- $f(x,x) > 0$ for this element a and
- $f(x',y')$ for all pairs $(x',y') \neq (x,x)$.

Different degrees are then simply associated with different values $v = f(x,x)$.

This construction provides us with degrees at each element $a \in U$. To get a general description of degrees, we need to relate the values corresponding to different elements $x, x' \in U$.

5.2°. Let us denote, by $f_{x,v}$, the morphism for which:

- $f_{x,v}(x,x) = v$ and
- $f_v(x',y') = 0$ for all pairs $(x',y') \neq (x,x)$.

We want, for every $a \neq b$, to connect the values v and w corresponding to functions $f_{x,v}$ and $f_{y,w}$. This connection comes from the following auxiliary result:

$$w \leq v \Leftrightarrow \exists f_{x\to y}\, \exists f_{y\to x}\, (f_{x\to y} \circ f_{x,v} \cdot f_{y\to x} = f_{y,w}).$$

Indeed, by definition of a composition, the values of the composition $g \circ f$ cannot exceed the largest value of each of the composed relations g and f. Thus, if $f_{x\to y} \circ f_{x,v} \cdot f_{y\to x} = f_{y,w}$, then the value $f_{y,w}(b,b) = w$ cannot exceed the maximum value v of the function $f_{x,v}$; thus, $w \leq v$.

Vice versa, if $w \leq v$, then we can take the following morphisms $f_{x\to y}$ and $f_{y\to x}$:

- $f_{x\to y}(x,y) = w$ and $f_{x\to y}(x',y') = 0$ for all other pairs $(x',y') \neq (x,y)$, and, similarly,
- $f_{y\to x}(y,x) = w$ and $f_{y\to x}(x',y') = 0$ for all other pairs $(x',y') \neq (y,x)$.

In this case, as one can easily check, we have $f_{x\to y} \circ f_{x,v} \cdot f_{y\to x} = f_{y,w}$.

5.3°. Now that we know how to describe the relation $w \leq v$ for functions $f_{x,v}$ and $f_{y,w}$ in ordered-category form, we can describe equality $v = w$ between the degrees v and w corresponding to morphisms $f_{x,v}$ and $f_{y,w}$ as $(v \leq w)\,\&\,(w \leq v)$, i.e., in view of Part 5.2, as:

$$(\exists f_{x\to y}\, \exists f_{y\to x}\, (f_{x\to y} \circ f_{x,v} \cdot f_{y\to x} = f_{y,w}))\,\&\,(\exists g_{y\to x}\, \exists g_{x\to y}\, (g_{y\to x} \circ f_{y,w} \cdot g_{x\to y} = f_{x,v})).$$

This enables us to identify degrees $v \in [0,1]$ in ordered-category terms – by identifying them with the functions $f_{x,v}$ and taking into account the above possibility to compare degrees at different elements a.

Hence, if we have an automorphism, degrees are mapped into degrees in a 1-1 and order-preserving way, i.e., indeed we have a monotonic bijection $H : [0,1] \to [0,1]$.

6°. To complete the proof, we need to show how, for each morphism f and for every two elements a and b, we can describe the value $f(x,y)$ in ordered-category terms. This will complete the proof that the given automorphism has the form $G_{\pi,H}$ for the mappings π and H as identified in Sections 4 and 5 of this proof.

6.1°. Let us first prove the following auxiliary result:

$$\exists f_{y\to x}\,(f_{y\to x}\circ f_{y,1}\circ f\circ f_{x,1}=f_{x,v})\Leftrightarrow v\le f(x,y).$$

Indeed, by definition of a composition, the composition $c\stackrel{\text{def}}{=}f\circ f_{x,1}$ has the following form:

- $c(x,y')=f(x,y')$ for all y' and
- $c(x',y')=0$ for all y' and for all $x'\neq a$.

Similarly, the composition $c'\stackrel{\text{def}}{=}f_{y,1}\circ f\circ f_{x,1}=f_{y,1}\circ c$ has the following form:

- $c'(x,y)=f(x,y)$, and
- $c'(x',y')=0$ for all other pairs $(x',y')\neq(x,y)$.

As we have argued in Part 5 of this proof, the value of a composition function cannot exceed the maximum value of each of the composed morphisms. Thus, for the composition $f_{y\to x}\circ f_{y,1}\circ f\circ f_{x,1}=f_{y\to x}\circ c'$, the maximum value cannot exceed the maximum value $f(x,y)$ of the morphism c'. Thus, if $f_{y\to x}\circ c'=f_{x,v}$, the maximum value v of the morphism $f_{x,v}$ cannot exceed $f(x,y)$: $v\le f(x,y)$.

Vice versa, for every $v\le f(x,y)$, we can construct a morphism $f_{y\to x}$ for which $f_{y\to x}\circ c'=f_{x,v}$: namely, we can take:

- $f_{y\to x}(y,x)=v$, and
- $f_{y\to x}(x',y')=0$ for all pairs $(x',y')\neq(y,x)$.

One can easily check that in this case indeed $f_{y\to x}\circ c'=f_{x,v}$.

6.2°. For each morphism f and for every two elements a and b, we can identify the degree $f(x,y)$ as the largest degree v for which the inequality $v\le f(x,y)$ holds.

Since, according to Part 6.1 of this proof, the inequality $v\le f(x,y)$ can be described in ordered-category terms, we can thus conclude that the degree $f(x,y)$ can also be described in ordered-category terms.

The proposition is proven.

4. Conclusions

Many concepts of modern mathematics, starting from the basic notions of sets and functions, are described in terms of category theory. many other mathematical concepts can be reformulated in category terms. Due to the general nature of category theory, such a reformulation often helps to extend notions and results from one area to different areas of mathematics.

Because of this potential advantage, it is reasonable to ask whether similar *fuzzy* notions can also be described in category terms. In this paper, we show that fuzzy relations – i.e., fuzzy analogues of functions – can indeed be described in category terms. Specifically, we show that, in the corresponding fuzzy category, we can describe both:

- elements of the original universe of discourse (modulo a 1-1 permutation), and
- fuzzy degrees (modulo a 1-1 monotonic mapping from the interval $[0,1]$ onto itself).

At this moment, what we have is a very theoretical paper. However, we hope that, similarly to how the reformulation of crisp notions in category terms can help generalize the corresponding results, our reformulation will help extend fuzzy results to more general situations – and thus, will facilitate future applications.

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