Bellman-Zadeh Fuzzy Optimization Under Interval Uncertainty

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Bellman-Zadeh fuzzy optimization. In many real-life situations, in addition to well-defined constraints that limit alternatives $x$ to a certain set $X$, we also have fuzzy constraints like “temperature should not be too high”. For such constraints, instead of knowing exactly which alternatives $x$ satisfy the desired constraint and which do not, we only have degree of confidence $(x) \in [0, 1]$ that describe to what extent the experts believe that the alternative $x$ satisfies the desired constraints; see, e.g., [2]. If we have an objective function $f(x)$ that we want to maximize, how do we optimize it under such fuzzy constraints?

A solution to this problem was proposed in a joint paper [1] by Richard Bellman of optimization fame and Lotfi Zadeh, father of fuzzy techniques: after selecting an “and”-operation $f_{\&}(a, b)$ – a function that is non-decreasing with respect to $a$ and $b$ – we should select an alternative $x$ that maximizes the expression

$$F(x) \overset{\text{def}}{=} f_{\&} \left( f(x) - f_+ - f_- ; \mu(x) \right),$$

(1)

where $f_- \overset{\text{def}}{=} \inf \{ f(y) : y \in X \}$ and $f_+ \overset{\text{def}}{=} \sup \{ f(y) : y \in X \}$.

Case of interval uncertainty. In the ideal case, we know the exact values of the objective function $f(x)$, and we know the exact values of the expert’s degree of confidence $\mu(x)$. In practice, we often only know $f(x)$ and $\mu(x)$ with interval uncertainty. In other words, for every $x$, we only know the bounds $f(x) \leq f(x) \leq \overline{f}(x)$ and $\underline{\mu}(x) \leq \mu(x) \leq \overline{\mu}(x)$ for these values.

Fuzzy optimization under interval uncertainty: formulation of the problem. For different values $f(x) \in [f(x), \overline{f}(x)]$ and $\mu(x) \in [\underline{\mu}(x), \overline{\mu}(x)]$, we get different values of $F(x)$.

In such situations, to make a decision, it is reasonable to find the range $[F(x), \overline{F}(x)]$ of possible values of $F(x)$. Once we have found this range, we can:
• either select all the alternatives which can be optimal for some \( F(x) \in [F(x), \overline{F}(x)] \), i.e., all alternatives for which \( F(x) \geq \sup_y F(y) \),

• or, if we want to select a single alternative, follow the usual Hurwicz decision-making strategy (see, e.g., [3]): find the value \( \alpha \in [0,1] \) that reflects the decision maker’s degree of optimism-pessimism, and select the alternative for which the value \( F_\alpha(x) \equiv \alpha \cdot \overline{F}(x) + (1 - \alpha) \cdot \underline{F}(x) \) is the largest possible.

**Main result.** We prove that

\[
F(x) = f_\&(\max(0, \frac{f(x) - f_+(x)}{\max(f(x), \overline{f}(x)) - f_+(x)}), \mu(x))
\]

and

\[
\overline{F}(x) = f_\&(\min(1, \frac{\overline{f}(x) - \min(\overline{f}(x), f_-(x))}{\max(\overline{f}(x), \underline{f}(x)) - \min(\overline{f}(x), f_-(x))}), \overline{\mu}(x))
\]

where

\[
f_-(x) \equiv \inf\{f(y) : y \in X, y \neq x\}, \quad f_+(x) \equiv \sup\{f(y) : y \in X, y \neq x\},
\]

\[
\overline{f}_-(x) \equiv \inf\{\overline{f}(y) : y \in X, y \neq x\}, \quad \overline{f}_+(x) \equiv \sup\{\overline{f}(y) : y \in X, y \neq x\}.
\]

**References**

