How to Detect Crisp Sets Based on Subsethood Ordering of Normalized Fuzzy Sets? How to Detect Type-1 Sets Based on Subsethood Ordering of Normalized Interval-Valued Fuzzy Sets?

Christian Servin
Computer Science and Information
Technology Systems Department
El Paso Community College
919 Hunter, El Paso, Texas 79915, USA
cservin@gmail.com

Olga Kosheleva and Vladik Kreinovich
University of Texas at El Paso
500 W. University
El Paso, TX 79968, USA
olgak@utep.edu, vladik@utep.edu

***Abstract*—If all we know about normalized fuzzy sets is which set is a subset of which, will we be able to detect crisp sets? It is known that we can do it if we allow all possible fuzzy sets, including non-normalized ones. In this paper, we show that a similar detection is possible if we only allow normalized fuzzy sets. We also show that we can detect type-1 fuzzy sets based on the subsethood ordering of normalized interval-valued fuzzy sets.**

I. Introduction

A fuzzy set is usually defined as function A from a certain set U – known as *Universe of discourse* – to the interval $[0,1]$; see, e.g., [1], [2], [3], [5], [6], [8]. Traditional – "crisp" – sets can be viewed as particular cases of fuzzy sets, for which $A(a) \in \{0,1\}$ for all x.

In most applications, we consider *normalized* fuzzy sets, i.e., fuzzy sets for which $A(x) = 1$ for some $x \in U$. For crisp sets, this corresponds to considering non-empty sets.

For two crisp sets, A is a subset or B if and only if $A(x) \leq B(x)$ for all x. The same condition is used as a definition of the subsethood ordering between fuzzy sets: a fuzzy set A is a *subset* of a fuzzy set B if $A(x) \leq B(x)$ for all x. Subsets $B \subseteq A$ which are different from the set A are called *proper* subsets of A.

A natural question is: if we have a class of all normalized fuzzy sets with the subsethood relation, can we detect which of these fuzzy sets are crisp? It is known that:

- if we alow *all* possible fuzzy sets – even non-normalized ones,
- then we can detect crisp sets; see, e.g., [7].

In this paper, we show that such a detection is possible even if we restrict ourselves only to normalized sets.

II. Main Result

In order to describe general crisp sets in terms of subsethood relation $\subseteq$ between fuzzy sets, we will first describe some auxiliary notions in these terms.

In this section, we only consider normalized fuzzy sets.

Proposition 1. *A normalized fuzzy set is a 1-element crisp set if and only if it has no proper normalized fuzzy subsets, i.e., if and only if $B \subseteq A$ implies $B = A$.*

Proof.

1°. Let us first prove that a 1-element crisp set $A = \{x_0\}$ (i.e., a set for which $A(x_0) = 1$ and $A(x) = 0$ for all $x \neq x_0$) has the desired property.

Indeed, if $B \subseteq A$, this means that $B(x) \leq A(x)$ for all x. For $x \neq x_0$, we have $A(x) = 0$, so we have $B(x) = 0$ as well.

Since B is a normalized fuzzy set, it has to attain value 1 somewhere. Since we have $B(x) = 0$ for all $x \neq x_0$, the only point $x \in U$ at which we can have $B(x) = 1$ is the point x_0. Thus, we have $B(x_0) = 1$.

So, indeed, we have $B(x) = A(x)$ for all x, i.e., $B = A$.

2°. Vice versa, let us prove that each normalized fuzzy set A which is different from a 1-element crisp set has a proper normalized fuzzy subset.

Indeed, since A, is normalized, we have $A(x_0) = 1$ for some x_0. Then, we can take $B = \{x_0\}$. Clearly, $B \subseteq A$, and, since A is not a 1-element crisp set, $B \neq A$.

The proposition is proven.

Definition 1. *By a* 2-element set*, we mean a normalized fuzzy set A for which $A(x) > 0$ for exactly two elements $x \in U$.*

Proposition 2. *For a normalized fuzzy set A which is not a 1-element crisp set, the following two conditions are equivalent to each other:*

- *A is a non-crisp 2-element set, and*
- *the class $\{B : B \subseteq A\}$ of all its subsets is linearly ordered, i.e.:*

$$\text{if } B_1 \subseteq A \text{ and } B_2 \subseteq A \text{ then } B_1 \subseteq B_2 \text{ or } B_2 \subseteq B_1.$$

Proof.

1°. Let us first prove that if A is a 2-element non-crisp set, then the class of all its subsets is linearly ordered.

Indeed, since A is a normalized fuzzy set, we must have $A(x_0) = 1$ for some $x_0 \in U$. Since A is a 2-element set, there must be one more value $x \in U$ for which $A(x) > 0$. Let us denote this value by x_1. So, we have:

- $A(x_0) = 1$,
- $A(x_1) > 0$ and
- $A(x) = 0$ for all other $x \in U$.

If we had $A(x_1) = 1$, then A would be a crisp set – namely, we would have $A = \{x_0, x_1\}$. Since A is a non-crisp set, we thus cannot have $A(x_1) = 1$, so we have $0 < A(x_1) < 1$.

If B is a normalized fuzzy set for which $B \subseteq A$, then for all x different from x_0 and x_1, we have $B(x) \le A(x) = 0$ and thus, $B(x) = 0$. Since B is normalized, we have $B(x) = 1$ for some x.

- This x cannot be different from x_0 and x_1 – because then $B(x) = 0$.
- This x cannot be equal to x_1, since then we would have $1 = B(x_1) \le A(x_1) < 1$ and $1 < 1$.

Thus, this x must be equal to x_0, i.e., we must have $B(x_0) = 1$. So, all fuzzy normalized subsets B of the set A have the following form:

- $B(x_0) = 1$,
- $B(x_1) \le A(x_1)$, and
- $B(x) = 0$ for all other x.

For two such subsets, we can have:

- either $B_1(x_1) \le B_2(x_1)$,
- or $B_2(x_1) \le B_1(x_1)$.

One can easily check that:

- if $B_1(x_1) \le B_2(x_1)$, then $B_1(x) \le B_2(x)$ for all x and thus, $B_1 \subseteq B_2$;
- similarly, if $B_2(x_1) \le B_1(x_1)$, then $B_2(x) \le B_1(x)$ for all x and thus, $B_2 \subseteq B_1$.

So, for every two normalized fuzzy subsets B_1 and B_2 of the set A, we have either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. Thus, the class of all such subsets is indeed linearly ordered.

2°. To complete the proof of Proposition 2, let us now prove that if a normalized fuzzy set A is not a 1-element fuzzy set and *not* a non-crisp 2-element set, then the class

$$\{B : B \subseteq A\}$$

is *not* linearly ordered, i.e., there exists normalized fuzzy subsets $B_1 \subseteq A$ and $B_2 \subseteq A$ for which $B_1 \not\subseteq B_2$ and $B_2 \not\subseteq B_1$.

The fact that the set A is not a 1-element set means that $A(x) > 0$ for at least two different values x.

By definition, a non-crisp 2-element set is a normalized fuzzy set:

- which is a 2-element set *and*
- which is not crisp.

So, if a normalized fuzzy set A is *not* a non-crisp 2-element set, this means that it is:

- either not a 2-element set
- or it is a crisp 2-element set.

Let us show that in both cases, we can find subsets $B_1 \subseteq A$ and $B_2 \subseteq A$ for which $B_1 \not\subseteq B_2$ and $B_2 \not\subseteq B_1$.

2.1°. Let us first consider the case when A is not a 2-element set, i.e., when, in addition to the point x_0 at which $A(x_0) = 1$, there exist at least two other points x_1 and x_2 for which $A(x_1) > 0$ and $A(x_1) > 0$.

In this case, we can take the following sets B_1 and B_2:

- $B_1(x_0) = B_2(x_0) = 1$;
- $B_1(x_1) = A(x_1)$ and $B_2(x_1) = 0$;
- $B_2(x_1) = 0$ and $B_2(x_2) = A(x_2)$, and
- $B_1(x) = B_2(x)$ for all other x.

One can see that $B_1(x) \le A(x)$ and $B_2(x) \le A(x)$ for all x, so indeed $B_1 \subseteq A$ and $B_2 \subseteq A$. However, here:

- $B_1(x_1) = A(x_1) > 0 = B_2(x_1)$, so we cannot have $B_1 \subseteq B_2$, because that would imply $B_1(x_1) \le B_2(x_1)$;
- similarly, $B_2(x_2) = A(x_2) > 0 = B_1(x_2)$, so we cannot have $B_2 \subseteq B_1$, because that would imply $B_2(x_2) \le B_1(x_2)$.

So, we indeed have $B_1 \not\subseteq B_2$ and $B_2 \not\subseteq B_1$.

2.2°. Let us now consider the case when A is a 2-element crisp set, i.e., when $A = \{x_0, x_1\}$.

In this case, we can take $B_1 = \{x_0\}$ and $B_2 = \{x_1\}$. Clearly, $B_1 \subseteq A$ and $B_2 \subseteq A$, but $B_1 \not\subseteq B_2$ and $B_2 \not\subseteq B_1$.

So, the proposition is proven.

Proposition 3. *A normalized fuzzy set A is a crisp 2-element set if and only if the following two conditions are satisfied:*

- *the set A itself is not a 1-element crisp set and not a 2-element non-crisp set, but*
- *each proper normalized fuzzy subsets $B \subseteq A$ is either a crisp 1-element sets or a non-crisp 2-element set.*

Proof.

1°. If A is a 2-element crisp set, i.e., if $A = \{x_0, x_1\}$ for some $x_0 \neq x_1$, then it is clearly:

- not a 1-element crisp set, and
- not a non-crisp 2-element set.

Let us prove that in this case, every proper normalized fuzzy subset $B \subseteq A$ is

- either a 1-element crisp set
- or a non-crisp 2-element set.

Since $A(x) > 0$ for only two values $x = x_0$ and $x = x_1$, and $B(x) \le A(x)$ for all x, the value $B(x)$ can be positive also for at most two values x_i.

If $B(x) > 0$ for only one value x, then, since B is normalized, for this x, we must have $B(x) = 1$. Thus, we have $B = \{x\}$, i.e., B is a 1-element crisp set.

If $B(x) > 0$ for two different values x, this means that we have $B(x_0) > 0$ and $B(x_1) > 0$. Since the set B is normalized, one of these value must be equal to 1. If the second one is equal to 1, we will have $B = A$ – but B is a proper subset. Thus, one of the values $B(x_i)$ is smaller than 1 – thus, B is a non-crisp 2-element set.

2°. Let us now prove that if a normalized fuzzy set A is not a 2-element crisp set, then one of the above properties is not satisfied, i.e.,

- either A is 1-element crisp set or a 2-element non-crisp set,
- or one of its proper subsets $B \subseteq A$ is *not* a non-crisp 2-element set.

In other words, we want to prove that if A is:

- not a crisp 1-element set,
- not a crisp 2-element set, and
- not a non-crisp 2-element set,

then one of its proper subsets $B \subseteq A$ is *not* a non-crisp 2-element set.

The condition on A means that it is:

- not a 1-element set and
- not a 2-element set.

This means that there must exist at least three different values $x \in U$ for which $A(x) > 0$. For one of these values, we have $A(x_0) = 1$, let us denote the other two values by x_1 and x_2, then $A(x_1) > 0$ and $A(x_2) > 0$.

Let us now take the following normalized fuzzy set B;

- $B(x_1) = 0.5 \cdot A(x_1)$,
- $B(x_2) = 0.5 \cdot A(x_2)$, and
- $B(x) = A(x)$ for all other x.

Here, $B(x_0) = A(x_0) = 1$, so B is indeed a normalized fuzzy set. One can easily check that $B(x) \le A(x)$ for all x, so it is indeed a subset of A. Since $A(x_1) > 0$, we have $B(x_1) = 0.5 \cdot A(x_1) \ne A(x_1)$, so B is a proper subset of A.

However, $B(x_0) = 1 > 0$, $B(x_1) > 0$, and $B(x_2) > 0$, so B is *not* a 2-element set.

The proposition is proven.

Comment. Now, we are ready to show that crisp sets can be described in terms of the subsethood relation.

Proposition 4. *A normalized fuzzy set is crisp if and only if we have one of the following two cases:*

- *A is a 1-element fuzzy set, or*
- *for every subset $B \subseteq A$ which is a non-crisp 2-element set, there exists a crisp 2-element set C for which*

$$B \subseteq C \subseteq A.$$

Comment. Since Propositions 1–3 show that the properties of being a crisp 1-element set, a crisp 2-element set, and a non-crisp 2-element set can all be described in terms of the subsethood relation, this Proposition shows that crispness can indeed be described in terms of subsethood.

Proof.

1°. Let us first prove that if A is a crisp set, then:

- either it is a 1-element crisp set,
- or for every non-crisp 2-element set $B \subseteq A$, there exists a crisp 2-element set C for which $B \subseteq C \subseteq A$.

Indeed, let B be a non-crisp 2-element set. This means that for some elements $x_0 \in U$ and $x_1 \in U$, we have:

- $B(x_0) = 1$,
- $0 < B(x_1) < 1$, and
- $B(x) = 0$ for all other x.

Since $B \subseteq A$, we have:

- $1 = B(x_0) \le A(x_0)$ – thus $A(x_0) = 1$; and
- $0 < B(x_2) \le A(x_1)$ – thus $A(x_1) > 0$.

The set A is crisp, so $A(x_1)$ can be either 0 or 1. Since $A(x_1) > 0$, we must have $A(x_1) = 1$. Thus, for a 2-element crisp set $C = \{x_0, x_1\}$, we have $B \subseteq C \subseteq A$.

2°. To complete our proof, let us prove that if a normalized crisp set A is *not* a crisp set, then there exists a non-crisp 2-element set $B \subseteq A$ for which no crisp 2-element set C satisfies the property $B \subseteq C \subseteq A$.

By definition, for a crisp set, all the values $A(x)$ are either 0s or 1s. So, the fact that A is not crisp means that we have $0 < A(x_1) < 1$ for some $x_1 \in U$.

Since A is normalized, there exists x_0 for which $A(x_0) = 1$. Let us now take the following set B;

- $B(x_0) = 1$,
- $0 < B(x_1) = A(x_1) < 1$, and
- $B(x) = 0$ for all other x.

Clearly, B is a non-crisp 2-element set a nd $B \subseteq A$.

If we had $B \subseteq C \subseteq A$ for some crisp 2-element set C, then due to $1 = B(x_0) \le C(x_0)$ and $B(x_1) \le C(x_1)$, we would have $C(x_0) = 1$ and $C(x_1) > 0$ – hence $C(x_1) = 1$ (since C is crisp). But in this case, $C(x_1) = 1 > A(x_1)$, so we cannot have $C \subseteq A$.

The proposition is proven.

III. Interval-Valued Case

Formulation of the problem. The traditional fuzzy logic assumes that experts can meaningfully describe their degrees of certainty by numbers from the interval $[0, 1]$. In practice, however, experts cannot meaningfully select a single number fuzzy describing their degree of certainty – since it is not possible to distinguish between, say, degrees 0.80 and 0.81. A more adequate description of the expert's uncertainty is when we allow to characterize the uncertainty by a whole range of possible numbers, i.e., by an interval $\left[\underline{A}(x), \overline{A}(x)\right]$.

This idea leads to *interval-valued* fuzzy numbers (see, e.g., [3], [4]), i.e., mappings that assign, to each element x from the Universe of discourse, an interval $A(x) = \left[\underline{A}(x), \overline{A}(x)\right]$.

For two interval-valued degrees $A = \left[\underline{A}, \overline{A}\right]$ and $B = \left[\underline{B}, \overline{B}\right]$, it is reasonable to say that $A \le B$ if

$$\underline{A} \le \underline{B} \text{ and } \overline{A} \le \overline{B}.$$

Thus, we can define a subsethood relation between two interval-valued fuzzy sets A and B as $A(x) \leq B(x)$ for all x.

An interval-valued fuzzy set is normalized if $\overline{A}(x_0) = 1$ for some x_0.

Traditional (*type-2*) fuzzy sets can be viewed as particular cases of interval-valued fuzzy sets, with "degenerate" intervals

$$[A(x), A(x)].$$

Here, we have a similar problem: can we detect traditional fuzzy sets based only on the subsethood relation between interval-valued fuzzy sets?

Let us show that this is indeed possible.

Definition 2. *By an* uncertain 1-element set*, we mean a normalized interval-valued fuzzy set A for which, for some $x_0 \in U$, we have:*

- $A(x_0) = [0, 1]$ *and*
- $A(x) = [0, 0]$ *for all other x.*

Proposition 5. *A normalized interval-valued fuzzy set A is an uncertain 1-element set if and only if it has no proper normalized subsets.*

Comment. So, we can determine uncertain 1-element sets based on the subsethood relation.

Proof.

1°. Let us first prove that for an uncertain 1-element set A, there are no proper subsets.

Indeed, if $A(x_0) = [0, 1]$, $A(x) = [0, 0]$ for all $x \neq x_0$, and $B(x) \leq A(x)$, then:

- for $x \neq x_0$, from $\underline{B}(x) \leq \underline{A}(x) = 0$ and $\overline{B}(x) \leq \overline{A}(x) = 0$, it follows that $\underline{B}(x) = \overline{B}(x) = 0$, so
$$B(x) = [0, 0] = A(x);$$
- for $x = x_0$, from $\underline{A}(x_0) \leq \overline{A}(x_0) = 0$, it follows that
$$\underline{B}(x_0) = 0 = \underline{A}(x_0).$$

On the other hand, B is a normalized interval-valued fuzzy set, so we must have $]overline B(x) = 1$ for some x. This cannot be for $x \neq x_0$, since then $\overline{B}(x) = 0$. So, the only remaining option is $x = x_0$. Hence, $\overline{B}(x_0) = 1$, thus, $\overline{B}(x_0) = \overline{A}(x_0)$.

Therefore, if $B \subseteq A$, then $B = A$. So, the normalized interval-valued fuzzy sets A does not have any proper subsets.

2°. To complete the proof, let us prove that if a normalized interval-valued fuzzy set has no proper subsets, then it is an uncertain 1-element set.

Indeed, since A is normalized, there exists an element x_0 for which $\overline{A}(x_0) = 1$. Then, as one can easily check, we have $B \subseteq A$, where:

- $B(x_0) = [0, 1]$, and
- $B(x) = [0, 0]$ for all other x

Since A has no proper subsets, we thus conclude that $A = B$, i.e., that A is an uncertain 1-element set.

The proposition is proven.

Definition 3. *By a* basic 1-element set*, we mean a normalized interval-valued fuzzy set A for which, for some $x_0 \in U$, we have:*

- $A(x_0) = [a, 1]$ *for some $a > 0$, and*
- $A(x) = [0, 0]$ *for all $x \neq x_0$.*

Definition 4. *By a* basic 2-element set*, we mean a normalized interval-valued fuzzy set A for which, for some $x_0 \neq x_1$, we have:*

- $A(x_0) = [0, 1]$*,*
- $A(x_1) = [0, a]$ *for some $a \in (0, 1)$, and*
- $A(x) = [0, 0]$ *for all other x.*

Proposition 6. *Let A be a normalized interval-valued fuzzy set which is not an uncertain 1-element set. Then, the following two conditions are equivalent to each other:*

- *the class $\{B : B \subseteq A\}$ of all subsets of A is linearly ordered;*
- *A is either a basic 1-element set or a basic 2-element set.*

Comment. So, we can determine, based on the subsethood relation, whether A is a basic set.

Proof.

1°. Let us first prove that if A is a basic 1-element set or a basic 2-element set, then the class of all its subsets is linearly ordered.

1.1°. Let us first consider the case when A is a basic 1-element set.

In this case, $B \subseteq A$ implies $\underline{B}(x) = \overline{B}(x) = 0$ for all $x \neq x_0$. Since B is normalized, then, similarly to the proof of Proposition 5, we get $\overline{B}(x_0) = 1$. The final inequality $\underline{B}(x_0) \leq \underline{A}(x_0) = a$ implies that for $b \stackrel{\text{def}}{=} \underline{B}(x_0)$, we have

$$b \leq a.$$

So, the set B has the following form:

- $B(x) = [0, 0]$ for all $x \neq x_0$, and
- $B(x_0) = [b, 1]$, where we denoted $b = \underline{B}(x_0)$.

One can easily check that the class of such sets is linearly ordered: namely, if for two such sets B_1 and B_2, we denote the corresponding values b by b_1 and b_2, then:

- if $b_1 \leq b_2$, then $B_1 \subseteq B_2$, and
- vice versa, if $b_2 \leq b_1$, then $B_2 \subseteq B_1$.

1.2°. Let us consider the case when A is a basic 2-element set.

Let $B \subseteq A$. Then, from $B(x) \leq A(x)$, we conclude that $B(x) = [0, 0]$ when x is different from x_0 and x_1, and that $\underline{B}(x_0) = \underline{B}(x_1) = 0$.

The set B is normalized, so $\overline{B}(x) = 1$ for some x.

- This x cannot be different from x_0 and x_1, since for such x, we have
$$\overline{B}(x) = 0 < 1.$$

- It cannot be equal to x_1, since we have

$$\overline{B}(x_1) \le \overline{A}(x_1) = a < 1.$$

Thus, the only possible element x is $x = x_0$, hence we have $\overline{B}(x_0) = 1$. The final inequality $\overline{B}(x_1) \le \overline{A}(x_1) = a$ implies that for $b \stackrel{\text{def}}{=} \overline{B}(x_1)$, we have $b \le a$.

So, the set B has the following form:

- $B(x) = [0, 0]$ for all x which are different from x_0 and x_1;
- $B(x_0) = [0, 1]$, and
- $B(x_1) = [0, b]$, where $b = \overline{B}(x_1)$.

One can easily check that the class of such sets is linearly ordered: namely, if for two such sets B_1 and B_2, we denote the corresponding values b by b_1 and b_2, then:

- if $b_1 \le b_2$, then $B_1 \subseteq B_2$, and
- vice versa, if $b_2 \le b_1$, then $B_2 \subseteq B_1$.

2°. Let us now prove that if the class of all normalized subsets of a normalized fuzzy interval-valued set A is linearly ordered, then A is either a basic 1-element set or a basic 2-element set.

Since the set A is normalized, there exists an element $x_0 \in U$ for which $\overline{A}(x_0) = 1$. Let us consider two possible cases:

- $\underline{A}(x_0) > 0$ and
- $\underline{A}(x_0) = 0$.

2.1°. Let us first consider he case when $\underline{A}(x_0) > 0$. Let us prove that in this case, we have a basic 1-element set, i.e., that $A(x) = [0, 0]$ for all $x \ne x_0$.

We will prove this by contradiction. Let us assume that $\overline{A}(x) > 0$ for some $x \ne x_0$. Then, we can consider the following two subsets of A:

- $B_1(x_0) = A(x_0)$, $B_2(x_0) = [0, 1]$;
- $B_2(x_1) = [0, 0]$, $B_2(x_1) = A(x_1)$, and
- $A(x) = B(x) = [0, 0]$ for al other $x \in U$.

One can easily check that $B_1 \subseteq A$ and $B_2 \subseteq A$. However:

- we have $\underline{B}_1(x_0) = \underline{A}(x_0) > 0 = \underline{B}_2(x_0)$, hence we cannot have $B_1 \subseteq B_2$;
- on the other hand, $\overline{B}_2(x_1) = \overline{A}(x_1) > 0 = \overline{B}_1(x_1)$, hence we cannot have $B_2 \subseteq B_1$.

The fact that here $B_1 \not\subseteq B_2$ and $B_2 \not\subseteq B_1$ shows that $\overline{A}(x) > 0$ is impossible. Thus, $\overline{A}(x) = 0$ for all $x \ne x_0$, so A is indeed a basic 1-element set.

2.2°. Let us first consider he case when $\underline{A}(x_0) = 0$. Let us prove that in this case, we have a basic 2-element set, i.e., that:

- $A(x_1) = [0, a]$ for some $x_1 \in U$ and some $a \in (0, 1)$, and
- $A(x) = [0, 0]$ for all other x.

Indeed, since $A(x_0) = [0, 1]$, but the set A is not an uncertain 1-element set, there exists some $x_1 \ne x_0$ for which $\overline{A}(x_1) > 0$.

2.2.1°. Let us prove that in this case, $A(x) = [0, 0]$ for all other x.

We prove this by contradiction. Let us assume that for some x_2, we have $x_2 \ne x_0$, $x_2 \ne x_1$ and $\overline{A}(x_2) > 0$. In this case, we can form the following two subsets B_1 and B_2;

- $B_1(x_0) = B_2(x_0) = [0, 1]$;
- $B_1(x_1) = A(x_1)$, $B_2(x_1) = [0, 0]$;
- $B_1(x_2) = [0, 0]$, $B_2(x_2) = A(x_2)$; and
- $B_1(x) = B_2(x) = [0, 0]$ or all other x.

Clearly, $B_1 \subseteq A$ and $B_2 \subseteq A$, but:

- $\overline{B}_1(x_1) > 0 = \overline{B}_2(x_1)$, so we cannot have $B_1 \subseteq B_2$, and
- $\overline{B}_2(x_2) = \overline{A}(x_2) > 0 = \underline{B}_1(x_2)$, so we cannot have

$$B_2 \subseteq B_1.$$

This contradicts to our assumption that the class of all subsets of A is linearly ordered. Thus, $A(x) = [0, 0]$ for all element x which are different from x_0 and x_1.

2.2.2°. Let us prove, by contradiction, that $\underline{A}(x_1) = 0$.

Indeed, if $\underline{A}(x_1) > 0$, then we can form the following sets B_1 and B_2:

- $B_1(x_0) = B_2(x_0) = [0, 1]$;
- $B_1(x_1) = \left[0, \overline{A}(x_1)\right]$, $B_2(x_1) = 0.5 \cdot \underline{A}(x_1)$.
- $B_1(x) = B_2(x) = [0, 0]$ for all other x.

One can easily check that $B_1 \subseteq A$ and $B_2 \subseteq A$, but:

- $\overline{B}_1(x_1) = \overline{A}(x_1) \ge \underline{A}(x_1) > 0.5 \cdot \underline{A}(x_1) = \overline{B}_2(x_1)$, so we do not have $B_1 \subseteq B_2$;
- on the other hand, $\underline{B}_2(x_1) = 0.5 \cdot \underline{A}(x_1) > 0 = \underline{B}_1(x_1)$, so we do not have $B_2 \subseteq B_1$ either.

This contradicts to our assumption that the class of all subsets of A is linearly ordered. This contradiction shows that

$$\underline{A}_1(x_1) = 0.$$

2.2.3°. Finally, let us prove that $\overline{A}(x_1) < 1$.

Indeed, if $\overline{A}(x_1) = 1$, i.e., if $A(x_1) = [0, 1]$, then we can find the following two sets $B_1 \subseteq A$ and $B_2 \subseteq A$ for which $B_1 \not\subseteq B_2$ and $B_2 \not\subseteq B_1$:

- $B_1(x_0) = [0, 1]$, $B_2(x_0) = [0, 0]$;
- $B_1(x_1) = [0, 0]$, $B_2(x_1) = A(x_1) = [0, 1]$, and
- $B_1(x) = B_2(x) = [0, 0]$ for all other x.

Then:

- $\overline{B}_1(x_0) = 1 > \overline{B}_2(x_0)$, so we cannot have $B_1 \subseteq B_2$;
- $\overline{B}_2(x_1) = 1 > 0 = \overline{B}_1(x_1)$, so we cannot have $B_2 \subseteq B_1$.

Contradiction show that we cannot have $\overline{A}(x_1) = 1$, thus

$$\overline{A}(x_1) < 1.$$

Thus, in this case, A is a basic 2-element set. The proposition is proven.

Proposition 7. *If A is a basic 1-element set or a basic 2-element set, then the following two properties are equivalent to each other:*

- *A is a crisp 1-element set;*
- *no proper superset of A is a basic 1-element set or a basic 2-element set.*

Comment. So, we can determine crisp 1-element sets based only on the subsethood relation.

Proof. If $A = \{x_0\}$, then clearly A cannot have any proper supersets which are basic 1-element or basic 2-element sets.

Vice versa, if A is a basic 1-element set with $\underline{A}(x_0) < 1$, then $B = \{x_0\}$ is its proper superset which is a a 1-element basic set.

Similarly, if A is a basic 2-element set, with $A(x_0) = [0, 1]$. $\underline{A}(x_1) = 0$, and $\overline{A}(x_1) < 1$, then we can have the following proper superset $B \supseteq A$ wit is also a basic 2-element set:

- $B(x_0) = [0, 1]$;
- $B(x_1) = \left[0, \dfrac{1 + \overline{A}(x_1)}{2}\right]$; and
- $B(x) = 0$ for all other x.

The proposition is proven.

Proposition 8. *For a normalized interval-valued fuzzy set, the following two conditions are satisfied:*

- *A is either an uncertain 1-element set or a basic 1-element set;*
- *A is a subset of a crisp 1-element set.*

Proof: straightforward.

Comment. Since we know how to describe, based on the subsethood relation,

- when A is an uncertain 1-element set, and
- when A is a basic set,

we can therefore determine:

- basic 1-element sets and
- basic 2-element sets

based on subsethood relation only.

Definition 5. *Let A be a basic 2-element set, with:*

- $A(x_0) = [0, 1]$,
- $A(x_1) = [0, a]$ *for some* $a \in (0, 1)$, *and*
- $A(x) = [0, 0]$ *for all other* x.

Then, by its type-1 cover*, we mean a normalized interval-valued fuzzy set A' for which:*

- $A'(x_0) = [1, 1]$,
- $A'(x_1) = [a, a]$, *and*
- $A'(x) = [0, 0]$ *for all other* x.

Let us show that the type-1 cover can be determined in terms of the subsethood relation.

Proposition 9. *Let A be a basic 2-element set. Then, its type-1 cover A' is the $\subseteq$-smallest normalized interval-valued fuzzy set that contains all the normalized interval-valued sets $B \supseteq A$ for which the following four conditions are satisfied:*

- *the set B is not a basic 2-element set;*
- *the class of all basic 2-element subsets of B is linearly ordered;*
- *the class $\{C : A \subseteq C \subseteq B\}$ of all normalized interval-valued set between A and B is linearly ordered; and*
- *the set B has only one uncertain 1-element subset.*

Proof.

1°. Let us first prove that a set B satisfies the above four conditions if and only if it has one the following two forms:

- either it has the form $B(x_0) = [b, 1]$ for some $b > 0$, $B(x_1) = A(x_1)$, and $B(x) = [0, 0]$ for all other x; we will call these B *of the first form*;
- or it has the form $B(x_0) = A(x_0)$, $B(x_1) = [b, a]$ for some $b > 0$, and $B(x) = [0, 0]$ for all other x; we will call these B *of the second form*.

1.1°. Let us first prove that the all the sets B of the first form satisfy all the above four conditions.

1.1.1°. Indeed, clearly, such B is not a basic 2-element set.

1.1.2°. If C is a basic 2-element set for which $C \subseteq B$, then we have:

- $C(x_0) = [0, 1]$,
- $C(x) = [0, 0]$ for all x different from x_0 and x_1, and
- $C(x_1) = [0, c]$ for some $c \le a$.

Clearly, the set of all such C is linearly ordered: if we have two such sets, corresponding to elements c_1 and c_2, then:

- if $c_1 \le c_2$, then we have $C_1 \subseteq C_2$, and
- if $c_2 \le c_1$, then we have $C_2 \subseteq C_1$.

1.1.3°. If $A \subseteq C \subseteq B$, then we have:

- $C(x_0) = [c, 1]$ for some $c \in [b, 1]$,
- $C(x_1) = A(x_1)$, and
- $C(x) = [0, 0]$ for all other x.

Thus, if we have two such sets, corresponding to elements c_1 and c_2, then:

- if $c_1 \le c_2$, then we have $C_1 \subseteq C_2$, and
- if $c_2 \le c_1$, then we have $C_2 \subseteq C_1$.

1.1.4°. Of course, the only uncertain 1-element set contained in B is the set corresponding to x_0.

All four conditions are proven.

1.2°. Let us now prove that the all the sets B of the second form satisfy all the above four conditions.

1.2.1°. Indeed, clearly, such B is not a basic 2-element set.

1.2.2°. If $C \subseteq B$ is a basic 2-element set, then we have:

- $C(x_0) = [0, 1]$,
- $C(x) = [0, 0]$ for all x different from x_0 and x_1, and
- $C(x_1) = [0, c]$ for some $c \le a$.

Clearly, the set of all such C is linearly ordered: if we have two such sets, corresponding to elements c_1 and c_2, then:

- if $c_1 \le c_2$, then we have $C_1 \subseteq C_2$, and

- if $c_2 \le c_1$, then we have $C_2 \subseteq C_1$.

$1.2.3^\circ$. If $A \subseteq C \subseteq B$, then we have:

- $C(x_0) = A(x_0)$,
- $C(x_1) = [c, a]$ for some $c \in [b, a]$, and
- $C(x) = [0, 0]$ for all other x.

Thus, if we have two such sets, corresponding to elements c_1 and c_2, then:

- if $c_1 \le c_2$, then we have $C_1 \subseteq C_2$, and
- if $c_2 \le c_1$, then we have $C_2 \subseteq C_1$.

$1.2.4^\circ$. Of course, the only uncertain 1-element set contained in B is the set corresponding to x_0.

All four conditions are proven.

1.3°. Let us now prove that if a set B satisfies the above four conditions, then B is either of the first form or of the second form.

$1.3.1^\circ$. Let us first prove that we must have $B(x) = [0, 0]$ for all elements x which are different from x_0 and x_1.

We will prove this by contradiction. Assume that $\overline{B}(x_2) > 0$ for some element x_2 which is different from x_0 and x_1. Then, in addition to a basic 2-element set $A \subseteq B$, we also have another basic 2-element set $C \subseteq B$ for which:

- $C(x_0) = [0, 1]$,
- $C(x_2) = \left[0, \overline{B}(x_2)\right]$, and
- $C(c) = [0, 0]$ for all other elements x.

Then:

- $\overline{A}(x_1) = a > 0 = \overline{C}(x_1)$, so we cannot have $A \subseteq C$; and
- $\overline{C}(x_2) > 0 = \overline{A}(x_2)$, so we cannot have $C \subseteq A$ either.

This contradicts to the condition that set of all basic 2-element sets which are subsets of B is linearly ordered.

Thus, $\overline{B}(x) > 0$ is impossible. So, indeed, $B(x) = [0, 0]$ for all elements x which are different from x_0 and x_1.

$1.3.2^\circ$. Due to Part 1.3.1 of this proof, the set B is uniquely described by its values $B(x_0)$ and $B(x_1)$. The condition that $A \subseteq B$ implies that $\overline{A}(x_0) = 1$ and that:

- $\overline{B}(x_0) \ge 0$,
- $\underline{B}(x_1) \ge 0$, and
- that $\overline{B}(x_1) \ge a = \overline{A}(x_1)$.

Since B is not a basic 2-element set and A is such a set, we have $B \ne A$. Thus, at least one of the above inequalities must be strict. Let us consider these three inequalities one by one.

$1.3.3^\circ$. Let us first consider the case when $\overline{B}(x_0) > 0$. Let us prove that in this case, we have $B(x_1) = A(x_1)$, i.e., that we have a set of the first form.

We will first prove, by contradiction, that $\underline{B}(x_1) = 0$. Indeed, if $\underline{B}(x_1) > 0$, then we can form the following two sets C_1 and C_2 for which $A \subseteq C_1 \subseteq B$, $A \subseteq C_2 \subseteq B$, but $C_1 \not\subseteq C_2$ and $C_2 \not\subseteq C_1$:

- $C_1(x_0) = A(x_0) = [0, 1]$, $C_1(x_1) = B(x_1)$, and $C_1(x) = [0, 0]$ for all other x;
- $C_2(x_0) = B(x_0)$, $C_2(x_1) = A(x_1)$, and $C_2(x) = [0, 0]$ for all other x.

Here:

- $\underline{C}_1(x_1) = \underline{B}(x_1) > 0 = \underline{C}_2(x_1)$, so we cannot have $$C_1 \subseteq C_2;$$
- $\underline{C}_2(x_0) = \underline{B}(x_0) > 0 = \underline{C}_1(x_0)$, so we cannot have $$C_2 \subseteq C_1.$$

This contradicts to our assumption that the class of all intermediate fuzzy sets C is linearly ordered. Thus, we must have $\underline{B}(x_1) = 0$.

Let us now prove, by contradiction, that $\overline{B}(x_1) = \overline{A}(x_1)$. Indeed, suppose that $\overline{B}(x_1) > \overline{A}(x_1)$. Then we can form the following two sets C_1 and C_2 for which $A \subseteq C_1 \subseteq B$, $A \subseteq C_2 \subseteq B$, but $C_1 \not\subseteq C_2$ and $C_2 \not\subseteq C_1$:

- $C_1(x_0) = A(x_0) = [0, 1]$, $C_1(x_1) = B(x_1)$, and $C_1(x) = [0, 0]$ for all other x;
- $C_2(x_0) = B(x_0)$, $C_2(x_1) = A(x_1)$, and $C_2(x) = [0, 0]$ for all other x.

Here:

- $\overline{C}_1(x_1) = \overline{B}(x_1) > \overline{A}(x_1) = \overline{C}_2(x_1)$, so we cannot have $$C_1 \subseteq C_2;$$
- $\underline{C}_2(x_0) = \underline{B}(x_0) > 0 = \underline{C}_1(x_0)$, so we cannot have $$C_2 \subseteq C_1.$$

This contradicts to our assumption that the class of all intermediate fuzzy sets C is linearly ordered. Thus, we must have
$$\overline{B}(x_1) = \overline{A}(x_1).$$

So, in this case, we indeed have a set of the first form.

$1.3.4^\circ$. Let us now consider the case when $\underline{B}(x_1) > 0$. Let us prove that in this case, we have $\overline{B}(x_0) = 0$ and $\overline{B}(x_1) = \overline{A}(x_1)$, i.e., that we have a set of the second form.

We will first prove, by contradiction, that $\underline{B}(x_0) = 0$. Indeed, if $\underline{B}(x_0) > 0$, then we can form the following two sets C_1 and C_2 for which $A \subseteq C_1 \subseteq B$, $A \subseteq C_2 \subseteq B$, but $C_1 \not\subseteq C_2$ and $C_2 \not\subseteq C_1$:

- $C_1(x_0) = A(x_0) = [0, 1]$, $C_1(x_1) = B(x_1)$, and $C_1(x) = [0, 0]$ for all other x;
- $C_2(x_0) = B(x_0)$, $C_2(x_1) = A(x_1)$, and $C_2(x) = [0, 0]$ for all other x.

Here:

- $\underline{C}_1(x_1) = \underline{B}(x_1) > 0 = \underline{C}_2(x_1)$, so we cannot have $$C_1 \subseteq C_2;$$
- $\underline{C}_2(x_0) = \underline{B}(x_0) > 0 = \underline{C}_1(x_0)$, so we cannot have $$C_2 \subseteq C_1.$$

This contradicts to our assumption that the class of all intermediate fuzzy sets C is linearly ordered. Thus, we must have

$$\underline{B}(x_0) = 0.$$

Let us now prove, by contradiction, that $\overline{B}(x_1) = \overline{A}(x_1)$. Indeed, suppose that $\overline{B}(x_1) > \overline{A}(x_1)$. Then we can form the following two sets C_1 and C_2 for which $A \subseteq C_1 \subseteq B$, $A \subseteq C_2 \subseteq B$, but $C_1 \not\subseteq C_2$ and $C_2 \not\subseteq C_1$:

- $C_1(x_0) = [0, 1]$, $C_1(x_1) = B(x_1)$, and $C_1(x) = [0, 0]$ for all other x;
- $C_2(x_0) = B(x_0)$, $C_2(x_1) = A(x_1)$, and $C_2(x) = [0, 0]$ for all other x.

Here:

- $\overline{C}_1(x_1) = \overline{B}(x_1) > \overline{A}(x_1) = \overline{C}_2(x_1)$, so we cannot have
$$C_1 \subseteq C_2;$$
- $\underline{C}_2(x_0) = \underline{B}(x_0) > 0 = \underline{C}_1(x_0)$, so we cannot have
$$C_2 \subseteq C_1.$$

This contradicts to our assumption that the class of all intermediate fuzzy sets C is linearly ordered. Thus, we must have

$$\overline{B}(x_1) = \overline{A}(x_1).$$

So, in this case, we indeed have a set of the second form.

1.3.5°. Finally, let us prove that the case when $\overline{B}(x_1) > \overline{A}(x_1)$ is not possible.

We will first prove, by contradiction, that in this case, $\underline{B}(x_0) = 0$. Indeed, if $\underline{B}(x_0) > 0$, then we can form the following two sets C_1 and C_2 for which $A \subseteq C_1 \subseteq B$, $A \subseteq C_2 \subseteq B$, but $C_1 \not\subseteq C_2$ and $C_2 \not\subseteq C_1$:

- $C_1(x_0) = A(x_0) = [0, 1]$, $C_1(x_1) = B(x_1)$, and $C_1(x) = [0, 0]$ for all other x;
- $C_2(x_0) = B(x_0)$, $C_2(x_1) = A(x_1)$, and $C_2(x) = [0, 0]$ for all other x.

Here:

- $\overline{C}_1(x_1) = \overline{B}(x_1) > \overline{A}(x_1) = \overline{C}_2(x_1)$, so we cannot have
$$C_1 \subseteq C_2;$$
- $\underline{C}_2(x_0) = \underline{B}(x_0) > 0 = \underline{C}_1(x_0)$, so we cannot have
$$C_2 \subseteq C_1.$$

This contradicts to our assumption that the class of all intermediate fuzzy sets C is linearly ordered. Thus, we must have

$$\underline{B}(x_0) = 0.$$

Let us now prove, by contradiction, that $\underline{B}(x_1) = 0$. Indeed, suppose that $\underline{B}(x_1) > 0$. Then we can form the following two sets C_1 and C_2 for which $A \subseteq C_1 \subseteq B$, $A \subseteq C_2 \subseteq B$, but $C_1 \not\subseteq C_2$ and $C_2 \not\subseteq C_1$:

- $C_1(x_0) = A(x_0) = [0, 1]$, $C_1(x_1) = \left[0, \overline{B}(x_1)\right]$, and $C_1(x) = [0, 0]$ for all other x;
- $C_2(x_0) = A(x_0) = [0, 1]$, $C_2(x_1) = \left[\underline{B}(x_1), \overline{A}(x_1)\right]$, and $C_2(x) = [0, 0]$ for all other x.

Here:

- $\overline{C}_1(x_1) = \overline{B}(x_1) > \overline{A}(x_1) = \overline{C}_2(x_1)$, so we cannot have
$$C_1 \subseteq C_2;$$
- $\underline{C}_2(x_1) = \underline{B}(x_1) > 0 = \underline{C}_1(x_1)$, so we cannot have
$$C_2 \subseteq C_1.$$

This contradicts to our assumption that the class of all intermediate fuzzy sets C is linearly ordered. Thus, we must have

$$\underline{B}(x_1) = 0.$$

Finally, $\overline{B}(x_1) < 1$, since otherwise B would have two uncertain 1-element subsets:

- a subset corresponding to x_0, and
- a subset corresponding to x_1,

Then, since we know that $\overline{B}(x_0) = 1$ and we have proved that $\underline{B}(x_0) = \underline{B}(x_1) = 0$ and $\overline{B}(x_1) < 1$, we conclude that the set B is a basic 2-element set – and we explicitly assumed that it is not.

Thus, the third inequality cannot be strict, so B is indeed either of the first form, or of the second form. Once can check that the smallest set containing all such sets is indeed the set A'.

The proposition is proven.

Definition 6. *Let A be an uncertain 1-element set, with:*

- $A(x_0) = [0, 1]$, *and*
- $A(x) = [0, 0]$ *for all other x.*

Then, by its type-1 cover, *we mean a crisp set $A' = \{x_0\}$.*

Proposition 10. *A normalized interval-valued fuzzy set is a type-1 set if and only if the following two conditions are satisfied:*

- *if $B \subseteq A$ for some uncertain 1-element set, then $B' \subset A$, and*
- *if $B \subseteq A$ for some basic 2-element set, then $B' \subseteq A$.*

Comment. Since we have shown that:

- the operation B',
- uncertain 1-element sets, and
- basic 2-element sets

can all be described in terms of the subsethood relation, we can thus conclude that we can detect type-1 sets based on the subsethood relation between normalized interval-valued fuzzy sets.

Proof.

1°. One can see that the type-1 cover of a set $A(x) = \left[\underline{A}(x), \overline{A}(x)\right]$ has the form $A'(x) = \left[\overline{A}(x), \overline{A}(x)\right]$.

For a type-1 set, $\underline{A}(x) = \overline{A}(x)$, thus $A' = A$, and clearly, $A \subseteq B$ implies $A' \subseteq B$.

2°. Vice versa, let us prove that if the above two conditions are satisfied, then A is a type-1 set, i.e., that $\underline{A}(x) = \overline{A}(x)$ for all x.

To prove this, let us consider two possible cases:

- elements x for which $\overline{A}(x) = 1$, and
- elements x for which $\overline{A}(x) < 1$.

2.1° Let us first consider an element x for which $\overline{A}(x) = 1$. In this case, $B \subseteq A$ for the uncertain 1-element set B for which $B(x) = [0, 1]$ and $B(y) = [0, 0]$ for all $y \neq x$. Then, $B' = \{x\}$, i.e., $B'(x) = [1, 1]$. Thus, from $B' \subseteq A$ it follows that $1 = \underline{B}'(x) \le \underline{A}(x)$, so $\underline{A}(x) = 1 = \overline{A}(x)$. So, for such elements x, we indeed have $\underline{A}(x) = \overline{A}(x)$.

2.2°. Finally, let us consider an element x for which $\overline{A}(x) < 1$. Since A is normalized, there exists an element x_0 for which $\overline{A}(x_0) = 1$. Now, we can form the following basic 2-element set B:

- $B(x_0) = [0, 1]$,
- $B(x) = \left[0, \overline{A}(x)\right]$, and
- $B(y) = [0, 0]$ for all other elements y.

Clearly, $B \subseteq A$, hence $B' \subseteq A$. Here, $B'(x) = \left[\overline{B}(x), \overline{B}(x)\right] = \left[\overline{A}(x), \overline{A}(x)\right]$. So, $B' \subseteq A$ implies $\underline{B}'(x) = \overline{A}(x) \le \underline{A}(x)$, thus $\underline{A}(x) = \overline{A}(x)$.

The proposition is proven.

Acknowledgments

This work was supported in part by the US National Science Foundation grant HRD-1242122.

The authors are thankful to all the participants of the 2017 Annual Conference of the North American Fuzzy Information Processing Society NAFIPS'2017 (Cancun, Mexico, October 16–18, 2017) for valuable suggestions.

References

[1] R. Belohlavek, J. W. Dauben, and G. J. Klir, *Fuzzy Logic and Mathematics: A Historical Perspective*, Oxford University Press, New York, 2017.

[2] G. Klir and B. Yuan, *Fuzzy Sets and Fuzzy Logic*, Prentice Hall, Upper Saddle River, New Jersey, 1995.

[3] J. M. Mendel, *Uncertain Rule-Based Fuzzy Systems: Introduction and New Directions*, Springer, Cham, Switzerland, 2017.

[4] J. M. Mendel and D. Wu, *Perceptual Computing: Aiding People in Making Subjective Judgments*, IEEE Press and Wiley, New York, 2010.

[5] H. T. Nguyen and E. A. Walker, *A First Course in Fuzzy Logic*, Chapman and Hall/CRC, Boca Raton, Florida, 2006.

[6] V. Novák, I. Perfilieva, and J. Močkoř, *Mathematical Principles of Fuzzy Logic*, Kluwer, Boston, Dordrecht, 1999.

[7] C. Servin, G. Muela, and V. Kreinovich, "Can we detect crisp sets based only on the subsethood ordering of fuzzy sets? fuzzy sets and/or crisp sets based on subsethood of interval-valued fuzzy sets?", *Proceedings of the 2017 Annual Conference of the North American Fuzzy Information Processing Society NAFIPS'2017*, Cancun, Mexico, October 16–18, 2017.

[8] L. A. Zadeh, "Fuzzy sets", *Information and Control*, 1965, Vol. 8, pp. 338–353.