

A New Kalman Filter Model for Nonlinear Systems Based on Ellipsoidal Bounding

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Abstract In this paper, a new filter model called set-membership Kalman filter for nonlinear state estimation problems was designed, where both random and unknown but bounded uncertainties were considered simultaneously in the discrete-time system. The main loop of this algorithm includes one prediction step and one correction step with measurement information, and the key part in each loop is to solve an optimization problem. The solution of the optimiza-

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tion problem produces the optimal estimation for the state, which is bounded by ellipsoids. The new filter was applied on a highly nonlinear benchmark example and a two-dimensional simulated trajectory estimation problem, in which the new filter behaved better compared with extended Kalman filter results. Sensitivity of the algorithm was discussed in the end.

Keywords Set-membership Kalman filter · State estimation · Ellipsoidal bounding · Nonlinear programming · Optimization methods

1 Introduction

State estimation is applicable to virtually all areas of engineering and science. Any discipline that is concerned with the mathematical modeling of its systems is a likely candidate for state estimation. This includes electrical engineering, mechanical engineering, chemical engineering, aerospace engineering, robotics, dynamical systems' control and many others. Nonlinear filtering can be a difficult and complex subject in the field of state estimation. It is certainly not as mature, cohesive, or well understood as linear filtering. There is still a lot of room for advances and improvement in nonlinear estimation techniques.

The optimal state estimation problem can be summarized as follows: given a mathematical model of a real system, and allowing some state perturbations and noise corrupted measurements, the state of the real system has to be estimated [1]. The estimation usually bases on the solving of an optimization problem, the estimated result relies on the assumptions made on uncertainties. Developed in the past hundreds years, the stochastic state estimation techniques are most widely applied in the real world. This approach bases on the probabilistic assumptions of the uncertainties in the system, such as Kalman filter [2] and extended Kalman filter (EKF) [3,4] where uncertain parts (usually noise) in the system are assumed to have certain probability distribution (usually Gaussian distribution).

However, in many cases these probability distributions could be questionable, especially when the real process generating the data are complex so that only simplified models can be practically used in the estimation process [5].

There is another interesting approach, referred to set-membership uncertainty state estimation. Developed since 1960s [6–8], this approach assumes that the uncertainty is unknown but bounded (UBB). No further assumption was made except for its membership of a given bound. Under this assumption, the optimal estimated state, noisy measurements and uncertainty are in some compact sets, respectively. This new technique is more appropriate in many cases where the bounded description is more realistic than stochastic distributed hypothesis. Classified by the geometrical representations, there are four major methods to bound the uncertainty, which are polytopes [9, 10], ellipsoids [11–13], zonotopes [14–16, 1, 17] and intervals [18–20]. Polytope can be used to obtain better estimated accuracy, however, one major drawback is its computation load in multi-dimensional nonlinear systems, especially to zonotope. Ellipsoid has been widely used due to its simplicity of propagation, but the Minkovski sum of two ellipsoids is not an ellipsoid anymore, therefore the prorogation of its related algorithm requires solving an optimization problem.

In this paper, a new filter model called set-membership Kalman filter (SKF) for nonlinear systems was designed, in which both random and set-membership uncertainties were considered at the same time. This work extends Benjamin Noack’s previous work in his PhD dissertation [21], where the linear case was discussed sufficiently. The novel SKF takes UBB uncertainties into account in both process equation and measurement equation, therefore it has a better uncertainty measures. It also keeps the recursive framework of random uncertainties from Kalman filter, thus the advantages of KF are reserved during the

prorogation process. A better estimation under these more reliable assumptions is calculated based on solving an optimization problem in each step.

Section 2 gives mathematics preliminaries and dynamical system which would be considered later. Section 3 shows the detailed derivation of this new filter model. Section 4 is the algorithm in a practical form. Section 5 demonstrates how this new filter model works and shows that the SKF behaves better than EKF in some cases. The last section is the conclusion and future work.

2 Mathematical Model

2.1 Preliminaries

The following definitions, theorems and corollaries are required for the derivation of the new filter model. The detailed proofs were given in [13].

Definition 2.1 Given S a positive-definite matrix, denoted by $S > 0$, a bounded ellipsoid \mathcal{E} in \mathbb{R}^n with nonempty interior is defined as

$$\mathcal{E} = \mathcal{E}(c, S) = \{x \in \mathbb{R}^n | (x - c)^T S^{-1} (x - c) \leq 1, S > 0\} \quad (1)$$

where $c \in \mathbb{R}^n$ is called the center of the ellipsoid \mathcal{E} , and S is the shape matrix which is positive-definite and specifies the size and orientation of the ellipsoid.

Definition 2.2 In geometry, the Minkowski sum is an operation of two sets A and B in Euclidean space \mathbb{R}^n , which is defined by adding each vector in A to each vector in B , i.e.,

$$A \oplus B = \{a + b | a \in A, b \in B\}. \quad (2)$$

Given K ellipsoids of \mathbb{R}^n

$$\mathcal{E}_k = \mathcal{E}(c_k, S_k), (k = 1, 2, \dots, K) \quad (3)$$

their Minkowski sum is

$$\mathcal{U}_K = \sum_{k=1}^K \mathcal{E}_k, \quad (4)$$

which is not an ellipsoid anymore but still a convex set.

Denote the problem of finding the smallest ellipsoid (under the criterion of matrix trace) containing the Minkowski sum of the K ellipsoids as Problem T^+ :

$$\mathcal{E}^* = \arg \min_{\mathcal{U}_K \subset \mathcal{E}} \text{tr} S \quad (\text{Problem } T^+), \quad (5)$$

and from [13], this ellipsoid \mathcal{E}^* exists and is unique.

Theorem 2.1 *The center of the optimal ellipsoid \mathcal{E}^* for Problem T^+ is given by*

$$c^* = \sum_{k=1}^K c_k \quad (6)$$

Theorem 2.2 *Let \mathcal{D} be the convex set of all vectors $\alpha \in \mathbb{R}^K$ with all $\alpha_k > 0$ and $\sum_{k=1}^K \alpha_k = 1$. For any $\alpha \in \mathcal{D}$, the ellipsoid $\mathcal{E}_\alpha = \mathcal{E}^+(c^*, S_\alpha)$, with c^* defined by (6) and*

$$S_\alpha = \sum_{k=1}^K \alpha_k^{-1} S_k, \quad (7)$$

contains \mathcal{U}_K .

Corollary 2.1 *Special case of Theorem (2.2). When $K = 2$, we have $\alpha_1 + \alpha_2 = 1$, the S_α can be rewritten as*

$$S_\alpha = \frac{1}{\alpha_1} S_1 + \frac{1}{\alpha_2} S_2 = (1 + \frac{1}{\beta}) S_1 + (1 + \beta) S_2 \quad (8)$$

where β can be any nonnegative real number.

Proof Let $\alpha_2 = \frac{1}{1+\beta}$, $\beta \geq 0$ one can easily get above result.

Theorem 2.3 *In the family $\mathcal{E}_\alpha = \mathcal{E}^+(c^*, S_\alpha)$, the minimal-trace ellipsoid containing the sum of the ellipsoids $\mathcal{E}_k = \mathcal{E}^+(c_k, S_k)$, $k = 1, 2, \dots, K$ is obtained for*

$$S_{\alpha^*} = \left(\sum_{k=1}^K \sqrt{\text{tr} S_k} \right) \left(\sum_{k=1}^K S_k \sqrt{\text{tr} S_k} \right) \quad (9)$$

Corollary 2.2 *Special case of Theorem (2.3). When $K = 2$, we have*

$$S_{\alpha^*} = \left(1 + \frac{1}{\beta^*}\right) S_1 + (1 + \beta^*) S_2 \quad (10)$$

where $\beta^* = \sqrt{\frac{\text{tr} S_1}{\text{tr} S_2}}$.

2.2 Dynamical System

Consider the following nonlinear dynamical system:

$$x_{k+1} = f_k(x_k, u_k, w_k, a_{1,k}, a_{2,k}, \dots, a_{I,k}) \quad (11a)$$

$$y_k = h_k(x_k, v_k, b_k) \quad (11b)$$

where x_k is a n -dimensional state vector, u_k is the known input vector, $w_k \sim \mathcal{N}(0, C_k^u)$ is a Gaussian system noise with covariance matrix C_k^u , $a_{i,k} \in \mathcal{E}(0, S_{ik}^u)$ is the unknown but bounded perturbation with shape matrix S_{ik}^u . $i = 1, 2, \dots, I$. denotes the i th set-membership perturbation in the prediction equation. $v_k \sim \mathcal{N}(0, C_k^z)$ is the a Gaussian measurement noise with covariance matrix C_k^z , and

$b_k \in \mathcal{E}(0, S_k^z)$ is the unknown but bounded perturbation with shape matrix S_k^z . In this literature, u and z in the parameters denote they are relative to system equation and measurement equation, respectively. All the notations above represent the information at time k .

The following Fig. 1 shows an estimated schematic diagram via set-membership Kalman filter in 2D case [16]. Different with standard Kalman filter, where the output is usually an gaussian distribution and the mean of the distribution was regarded as the estimated point, in set-membership Kalman filter, a set containing all the mean values of possible distributions was put out.

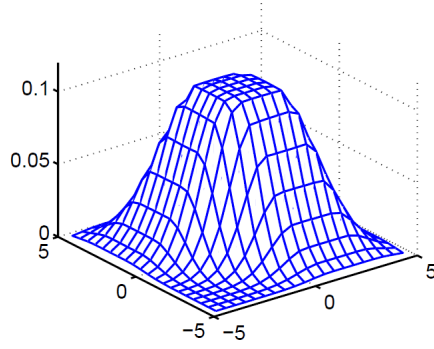


Fig. 1: Schematic diagram of 2D estimated result under SKF

2.3 Linearization

Recall the process of EKF, linearization is the first step in estimation for nonlinear dynamical systems. Perform Taylor series expansion for system equa-

tion (11a) around the point $(x_k = \hat{x}_k^+, u_k = u_k, w_k = 0, a_{i,k} = 0, 1 \leq i \leq I)$:

$$\begin{aligned}
x_{k+1} &= f_k(\hat{x}_k^+, u_k, 0, 0) + \frac{\partial f_k}{\partial x_k} \bigg|_{(\hat{x}_k^+, u_k, 0, 0)} (x_k - \hat{x}_k^+) + \frac{\partial f_k}{\partial w_k} \bigg|_{(\hat{x}_k^+, u_k, 0, 0)} w_k \\
&\quad + \sum_{i=1}^I \frac{\partial f_k}{\partial a_i} \bigg|_{(\hat{x}_k^+, u_k, 0, 0)} a_{i,k} + \dots \\
&\approx f_k(\hat{x}_k^+, u_k, 0, 0) + F_{x,k}(x_k - \hat{x}_k^+) + F_{w,k}w_k + \sum_{i=1}^I F_{a_i,k}a_{i,k} \quad (12) \\
&= F_{x,k}x_k + [f_k(\hat{x}_k^+, u_k, 0, 0) - F_{x,k}\hat{x}_k^+] + F_{w,k}w_k + \sum_{i=1}^I F_{a_i,k}a_{i,k} \\
&= F_{x,k}x_k + \tilde{u}_k + F_{w,k}w_k + \sum_{i=1}^I F_{a_i,k}a_{i,k}.
\end{aligned}$$

Here $\tilde{u}_k = f_k(\hat{x}_k^+, u_k, 0, 0) - F_{x,k}\hat{x}_k^+$.

Take Taylor series expansion for measurement equation (11b) around point

$(x_k = \hat{x}_k^-, v_k = 0, b_k = 0)$:

$$\begin{aligned}
y_k &= h_k(\hat{x}_k^-, 0, 0) + \frac{\partial h_k}{\partial x_k} \bigg|_{(\hat{x}_k^-, 0, 0)} (x_k - \hat{x}_k^-) + \frac{\partial h_k}{\partial v_k} \bigg|_{(\hat{x}_k^-, 0, 0)} v_k \\
&\quad + \frac{\partial h_k}{\partial b_k} \bigg|_{(\hat{x}_k^-, 0, 0)} b_k + \dots \quad (13) \\
&\approx h_k(\hat{x}_k^-, 0, 0) + H_{x,k}(x_k - \hat{x}_k^-) + H_{v,k}v_k + H_{b,k}b_k \\
&= H_{x,k}x_k + \tilde{z}_k + H_{v,k}v_k + H_{b,k}b_k.
\end{aligned}$$

Here $\tilde{z}_k = h_k(\hat{x}_k^-, 0, 0) - H_{x,k}\hat{x}_k^-$. $\tilde{z}_k = 0$ if measurement equation is linear.

Then we get the a linearized system for the original system (11)

$$x_{k+1} = F_{x,k}x_k + \tilde{u}_k + F_{w,k}w_k + A_k \quad (14a)$$

$$y_k = H_{x,k}x_k + \tilde{z}_k + H_{v,k}v_k + H_{b,k}b_k \quad (14b)$$

where $A_k = \sum_{i=1}^I F_{a_i,k}a_{i,k}$.

Both priori estimation \hat{x}_k^- and posteriori estimation \hat{x}_k^+ are random variables. Assume that the expectation and covariance matrix of priori estimation

\hat{x}_k^- are $\hat{\mu}_k^-$ and C_k^- , the expectation and covariance matrix of posteriori estimation \hat{x}_k^+ are $\hat{\mu}_k^+$ and C_k^+ . All the priori expectations $\hat{\mu}_k^-$ form an ellipsoid centered at \hat{x}_k^{c-} with shape matrix S_k^- , i.e., $\hat{\mu}_k^- \in \mathcal{E}(\hat{x}_k^{c-}, S_k^-)$. Similarly to posteriori expectation we have $\hat{\mu}_k^+ \in \mathcal{E}(\hat{x}_k^{c+}, S_k^+)$.

Our objective is to calculate the explicit expressions of \hat{x}_k^{c-} , C_k^- , S_k^- and \hat{x}_k^{c+} , C_k^+ , S_k^+ .

3 Derivation of Set-membership Kalman Filter

After linearization of the nonlinear dynamical system (11), in this section we derive the set-membership Kalman filter model. Conclusions from section 2.1 are required and the results of this section would be summarized into one algorithm in section 4.

3.1 Prediction

Assume that the difference between the true state x_k and the posteriori estimations center \hat{x}_k^{c+} contains two components, i.e., the random part and the UBB part:

$$x_k - \hat{x}_k^{c+} = \tilde{x}_k^{r+} + \tilde{x}_k^{s+}. \quad (15)$$

So from last section we can get $\tilde{x}_k^{r+} \sim N(0, C_k^+)$ and $\tilde{x}_k^{s+} \in \mathcal{E}(0, S_k^+)$. And the mean squared error of posteriori estimation is given by

$$\begin{aligned} E[(x_k - \hat{x}_k^{c+})(x_k - \hat{x}_k^{c+})^T] &= E[(\tilde{x}_k^{r+} + \tilde{x}_k^{s+})(\tilde{x}_k^{r+} + \tilde{x}_k^{s+})^T] \\ &= E[\tilde{x}_k^{r+} \tilde{x}_k^{r+,T}] + E[\tilde{x}_k^{s+} \tilde{x}_k^{s+,T}] = C_k^+ + \tilde{x}_k^{s+} \tilde{x}_k^{s+,T}. \end{aligned} \quad (16)$$

Recalling EKF we have

$$\hat{x}_{k+1}^- = F_{x,k}\hat{x}_k^+ + \tilde{u}_k + F_{w,k}w_k + A_k. \quad (17)$$

Notice that $F_{w,k}w_k \sim \mathcal{N}(0, F_{w,k}C_k^u F_{w,k}^T)$, so for a fixed posteriori estimation $\hat{\mu}_k^+ \in \mathcal{E}(\hat{x}_k^{c+}, S_k^+)$, the predicted state follows by

$$\hat{x}_{k+1}^- = F_{x,k}\hat{\mu}_k^+ + \tilde{u}_k + A_k + F_{w,k}w_k \sim \mathcal{N}(F_{x,k}\hat{\mu}_k^+ + \tilde{u}_k + A_k, F_{w,k}C_k^u F_{w,k}^T | \hat{x}_k^+). \quad (18)$$

Therefore the expectation of \hat{x}_{k+1}^- would be

$$\hat{\mu}_{k+1}^- = \mathbb{E}(\hat{x}_{k+1}^-) = F_{x,k}\hat{\mu}_k^+ + \tilde{u}_k + A_k, \quad (19)$$

which forms a set $\mathcal{E}(\hat{x}_{k+1}^{c-}, S_{k+1}^-)$ when \hat{x}_k^+ being ergodic in the set $\mathcal{E}(\hat{x}_k^{c+}, S_k^+)$.

Without loss of generality we have

$$\hat{x}_{k+1}^{c-} = F_{x,k}\hat{x}_k^{c+} + \tilde{u}_k. \quad (20)$$

Then the difference between the true state and the priori estimation center would be

$$\begin{aligned} x_{k+1} - \hat{x}_{k+1}^{c-} &= F_{x,k}(x_k - \hat{x}_k^{c+}) + F_w w_k + A_k \\ &= F_{x,k}(\tilde{x}_k^{r+} + \tilde{x}_k^{s+}) + F_w w_k + A_k. \end{aligned} \quad (21)$$

Consider its covariance matrix we have

$$\begin{aligned}
& \mathbb{E}[(x_{k+1} - \hat{x}_{k+1}^{c-})(x_{k+1} - \hat{x}_{k+1}^{c-})^T] \\
&= \mathbb{E}\{[F_{x,k}(\tilde{x}_k^{r+} + \tilde{x}_k^{s+}) + F_w w_k + A_k] \cdot [F_{x,k}(\tilde{x}_k^{r+} + \tilde{x}_k^{s+}) + F_w w_k + A_k]^T\} \\
&= F_{x,k} \mathbb{E}[(\tilde{x}_k^{r+} + \tilde{x}_k^{s+})(\tilde{x}_k^{r+} + \tilde{x}_k^{s+})^T] F_{x,k}^T + F_{x,k} \tilde{x}_k^{s+} A_k^T \\
&\quad + F_{w,k} \mathbb{E}(w_k w_k^T) F_{w,k}^T + A_k \hat{x}_k^{s+,T} + A_k A_k^T \\
&= F_{x,k} C_k^+ F_{x,k}^T + F_{w,k} C_k^u F_{w,k}^T + (F_{x,k} \tilde{x}_k^{s+} + A_k)(F_{x,k} \tilde{x}_k^{s+} + A_k)^T.
\end{aligned} \tag{22}$$

Compared to equation (32), we find that the predicted random uncertainty can be represented by

$$C_{k+1}^- = F_{x,k} C_k^+ F_{x,k}^T + F_{w,k} C_k^u F_{w,k}^T. \tag{23}$$

Notice that a possible posteriori mean value $\hat{x}_k^+ \in \mathcal{E}(\hat{x}_k^{c+}, S_k^+)$, and

$$A_k = \sum_{i=1}^I F_{ai} a_{i,k}, a_{i,k} \in \mathcal{E}(0, S_{i,k}^u) \tag{24}$$

$$F_{ai} a_{i,k} \in \mathcal{E}(0, F_{ai} S_{i,k}^u F_{ai}^T). \tag{25}$$

So

$$A_k \in \sum_{i=1}^I \mathcal{E}(0, F_{ai} S_{i,k}^u F_{ai}^T). \tag{26}$$

i.e., A_k is one fixed element of a convex set which is the Minkowski sum of I ellipsoids.

Recalling (19) we have

$$\begin{aligned}
\hat{\mu}_{k+1}^- &= \mathbb{E}(\hat{x}_{k+1}^-) = F_{x,k} \hat{\mu}_k^+ + \tilde{u}_k + A_k \\
&\in \mathcal{E}(F_{x,k} \hat{x}_k^{c+} + \tilde{u}_k, F_{x,k} S_k^+ F_{x,k}^T) \oplus \sum_{i=1}^I \mathcal{E}(0, F_{ai} S_{i,k}^u F_{ai}^T)
\end{aligned} \tag{27}$$

Recalling 2.3, there exists an optimal ellipsoid $\mathcal{E}(c_k^*, S_{\alpha^*, k})$ such that

$$\begin{aligned} F_{x,k} \hat{\mu}_k^+ + \tilde{u}_k + A_k &\in \mathcal{E}(F_{x,k} \hat{x}_k^{c+} + \tilde{u}_k, F_{x,k} S_k^+ F_{x,k}^T) \oplus \sum_{i=1}^I \mathcal{E}(0, F_{ai} S_{i,k}^u F_{ai}^T) \\ &\subset \mathcal{E}(\hat{x}_{k+1}^{c-}, S_{k+1}^-) \end{aligned} \quad (28)$$

From 2.1 we can get the center of the ellipsoid:

$$\hat{x}_{k+1}^{c-} = F_{x,k} \hat{x}_k^{c+} + \tilde{u}_k. \quad (29)$$

From 2.3 we can calculate the shape matrix of the ellipsoid:

$$\begin{aligned} S_{k+1}^- &= (\sqrt{\text{tr}(F_{x,k} S_k^+ F_{x,k}^T)} + \sum_{i=1}^I \sqrt{\text{tr}(F_{a,i} S_{i,k}^u F_{a,i}^T)}) \\ &\quad \cdot \left(\frac{F_{x,k} S_k^+ F_{x,k}^T}{\sqrt{\text{tr}(F_{x,k} S_k^+ F_{x,k}^T)}} + \sum_{i=1}^I \frac{F_{a,i} S_{i,k}^u F_{a,i}^T}{\sqrt{\text{tr}(F_{a,i} S_{i,k}^u F_{a,i}^T)}} \right) \end{aligned} \quad (30)$$

Equation (23), (29) and (30) gave us the elicited expressions of C_k^- , \hat{x}_k^{c-} and S_k^- respectively.

3.2 Filtering

Similar with (15), here we assume that

$$x_k - \hat{x}_k^{c-} = \tilde{x}_k^{r-} + \tilde{x}_k^{s-}. \quad (31)$$

So from last section we can get $\tilde{x}_k^{r-} \sim N(0, C_k^-)$ and $\tilde{x}_k^{s-} \in \mathcal{E}(0, S_k^-)$. And the mean squared error of priori estimation is given by

$$\begin{aligned} \mathbb{E}[(x_k - \hat{x}_k^{c-})(x_k - \hat{x}_k^{c-})^T] &= \mathbb{E}[(\tilde{x}_k^{r-} + \tilde{x}_k^{s-})(\tilde{x}_k^{r-} + \tilde{x}_k^{s-})^T] \\ &= \mathbb{E}[\tilde{x}_k^{r-} \tilde{x}_k^{r-,T}] + \mathbb{E}[\tilde{x}_k^{s-} \tilde{x}_k^{s-,T}] = C_k^- + \tilde{x}_k^{s-} \tilde{x}_k^{s-,T}. \end{aligned} \quad (32)$$

$$\tilde{z}_k = h_k(\hat{x}_k^-, 0, 0) - H_{x,k}\hat{x}_k^- \quad (33)$$

$$y_k - \tilde{z}_k = H_{x,k}x_k + H_{v,k}v_k + H_{b,k}b_k. \quad (34)$$

Therefore, recalling equation (13) and (14) and the derivation process in EKF, we also assume

$$\begin{aligned} \hat{x}_k^+ &= \hat{x}_k^- + K_k[y - h_k(\hat{x}_k^-, 0, 0)] = \hat{x}_k^- + K_k[y - \tilde{z}_k(\hat{x}_k^-) - H_{x,k}\hat{x}_k^-] \\ &= (I - K_k H_{x,k})\hat{x}_k^- + K_k[y - \tilde{z}_k(\hat{x}_k^-)]. \end{aligned} \quad (35)$$

The expectations $\hat{\mu}_k^+$ of posteriori estimations \hat{x}_k^+ would be

$$\hat{\mu}_k^+ = E(\hat{x}_k^+) = (I - K_k H_{x,k})\hat{\mu}_k^- + K_k[y - \tilde{z}_k(\hat{\mu}_k^-)]. \quad (36)$$

The center of the ellipsoid $\mathcal{E}(\hat{x}_k^{c+}, S_k^+)$ would be

$$\begin{aligned} \hat{x}_k^{c+} &= \hat{x}_k^{c-} + K_k[y - h_k(\hat{x}_k^{c-}, 0, 0)] = \hat{x}_k^{c-} + K_k[y - \tilde{z}_k(\hat{x}_k^{c-}) - H_{x,k}\hat{x}_k^{c-}] \\ &= (I - K_k H_{x,k})\hat{x}_k^{c-} + K_k[y - \tilde{z}_k(\hat{x}_k^{c-})]. \end{aligned} \quad (37)$$

Subtract \hat{x}_k^{c+} from the true state x_k we get:

$$\begin{aligned} x_k - \hat{x}_k^{c+} &= x_k - (I - K_k H_{x,k})\hat{x}_k^{c-} - K_k[y - \tilde{z}_k(\hat{x}_k^{c-})] \\ &= x_k - (I - K_k H_{x,k})\hat{x}_k^{c-} - K_k(H_{x,k}x_k + H_{v,k}v_k + H_{b,k}b_k) \\ &= (I - K_k H_{x,k})(\tilde{x}_k^{r-} + \tilde{x}_k^{s-}) - K_k(H_{v,k}v_k + H_{b,k}b_k). \end{aligned} \quad (38)$$

So the mean squared error of the posteriori estimation center would be

$$\begin{aligned} &E[(x_k - \hat{x}_k^{c+})(x_k - \hat{x}_k^{c+})^T] \\ &= E\{[(I - K_k H_{x,k})(\tilde{x}_k^{r-} + \tilde{x}_k^{s-}) + K_k(H_{v,k}v_k + H_{b,k}b_k)] \cdot \\ &\quad [(I - K_k H_{x,k})(\tilde{x}_k^{r-} + \tilde{x}_k^{s-}) + K_k(H_{v,k}v_k + H_{b,k}b_k)]^T\} \\ &= (I - K_k H_{x,k})C_k^- (I - K_k H_{x,k})^T + K_k H_{v,k} C_k^z H_{v,k}^T K_k^T \\ &\quad + [(I - K_k H_{x,k})\tilde{x}_k^{s-} - K_k H_{b,k}b_k][(I - K_k H_{x,k})\tilde{x}_k^{s-} - K_k H_{b,k}b_k]^T. \end{aligned} \quad (39)$$

Compared with equation (32), we get

$$C_k^+ = (I - K_k H_{x,k}) C_k^- (I - K_k H_{x,k})^T + K_k H_{v,k} C_k^z H_{v,k}^T K_k^T. \quad (40)$$

Similar to Kalman filter (KF), the covariance matrices in the SKF provide us with a measure for uncertainty in our predicted and filtering state estimate, which is a very important feature for various applications of filtering theory since we then know how much to trust our predictions and estimates.

Notice that

$$\hat{\mu}_k^- \in \mathcal{E}(\hat{x}_k^{c-}, S_k^-) \quad (41)$$

and

$$y_k - \tilde{z}_k = H_{x,k} x_k + H_{v,k} v_k + H_{b,k} b_k \in \mathcal{E}(H_{x,k} x_k + H_{v,k} v_k, H_{b,k} S_k^z H_{b,k}^T). \quad (42)$$

So back to equation (36) we have

$$\begin{aligned} \hat{\mu}_k^+ &= (I - K_k H_{x,k}) \hat{\mu}_k^- + K_k (y_k - \tilde{z}_k) \\ &\in (I - K_k H_{x,k}) \mathcal{E}(\hat{x}_k^{c-}, S_k^-) \oplus K_k \mathcal{E}(H_{x,k} x_k + H_{v,k} v_k, H_{b,k} S_k^z H_{b,k}^T) \\ &= \mathcal{E}[(I - K_k H_{x,k}) \hat{x}_k^{c-}, (I - K_k H_{x,k}) S_k^- (I - K_k H_{x,k})^T] \\ &\quad \oplus \mathcal{E}[K_k (H_{x,k} x_k + H_{v,k} v_k), K_k H_{b,k} S_k^z H_{b,k}^T K_k^T] \subset \mathcal{E}(\hat{x}_k^{c+}, S_k^+), \end{aligned} \quad (43)$$

where the midpoint is exactly in accordance with our previous assumption (37):

$$\hat{x}_k^{c+} = (I - K_k H_{x,k}) \hat{x}_k^{c-} + K_k [y - \tilde{z}_k(\hat{x}_k^{c-})], \quad (44)$$

and from Corollary 2.1 we have

$$S_k^+(\beta) = (1 + \frac{1}{\beta})(I - K_k H_{x,k}) S_k^- (I - K_k H_{x,k})^T + (1 + \beta) K_k H_{b,k} S_k^z H_{b,k}^T K_k^T. \quad (45)$$

3.3 Optimization Problem

Now comparing to its counterpart in EKF, the only thing left is to derive the new optimal Kalman gain, which should minimize the mean square error of the posteriori estimation.

Here we introduce another parameter $\eta \in [0, 1]$ to balance the random uncertainty and set-membership in the dynamical system, and define the following cost function as:

$$J(\beta) = (1 - \eta)\text{tr}(C_k^+) + \eta\text{tr}(S_k^+(\beta)) \quad (46)$$

which represents the global uncertainty of the posteriori estimation. The new optimal Kalman gain to be found should be used to minimize this cost function in a comprehensive way.

Plugging (32) and (45) into (46) we get:

$$\begin{aligned} J(\beta) = & (1 - \eta)\text{tr}[(I - K_k H_{x,k})C_k^- (I - K_k H_{x,k})^T] \\ & + (1 - \eta)\text{tr}[K_k H_{v,k} C_k^z H_{v,k}^T K_k^T] \\ & + \eta(1 + \frac{1}{\beta})\text{tr}[(I - K_k H_{x,k})S_k^- (I - K_k H_{x,k})^T] \\ & + \eta(1 + \beta)\text{tr}(K_k H_{b,k} S_k^z H_{b,k}^T K_k^T) \\ \triangleq & (1 - \eta)\text{tr}[(I - K_k H_{x,k})C_k^- (I - K_k H_{x,k})^T] \\ & + (1 - \eta)\text{tr}[K_k H_{v,k} C_k^z H_{v,k}^T K_k^T] + \eta(1 + \frac{1}{\beta})M + \eta(1 + \beta)N. \end{aligned} \quad (47)$$

where M and N are defined directly from above.

Notice that the cost function J relies on two variables K_k and β . Firstly we minimize J respect with β .

Since $M > 0$ and $N > 0$, therefore

$$(1 + \frac{1}{\beta})M + (1 + \beta)N = M + N + \frac{1}{\beta}M + \beta N \geq M + N + 2\sqrt{MN}. \quad (48)$$

When $\frac{1}{\beta}M = \beta N$, i.e., $M = \beta^2 N$, $\beta = \beta_1 = \sqrt{\frac{M}{N}}$, we have

$$(1 + \frac{1}{\beta_1})M + (1 + \beta_1)N = M + N + 2\sqrt{MN} = (\sqrt{M} + \sqrt{N})^2 \quad (49)$$

Therefore we can find the local minimum point of function J with respect to β :

$$\begin{aligned} J(\beta_1) = & (1 - \eta)\text{tr}[(I - K_k H_{x,k})C_k^-(I - K_k H_{x,k})^T] \\ & + (1 - \eta)\text{tr}[K_k H_{v,k}C_k^z H_{v,k}^T K_k^T] + \eta(\sqrt{M} + \sqrt{N})^2. \end{aligned} \quad (50)$$

Next we calculate the global minimum by taking K_k into account.

Notice that

$$\begin{aligned} \frac{\partial \sqrt{M}}{\partial K} &= \frac{1}{2}M^{-\frac{1}{2}} \frac{\partial M}{\partial K} = -\frac{1}{2\sqrt{M}}(I - K_k H_{x,k})(S_k^{-,T} + S_k^-)H_{x,k}^T \\ &= -\frac{1}{\sqrt{M}}(I - K_k H_{x,k})S_k^- H_{x,k}^T \end{aligned} \quad (51)$$

$$\begin{aligned} \frac{\partial \sqrt{N}}{\partial K} &= \frac{1}{2}N^{-\frac{1}{2}} \frac{\partial N}{\partial K} = \frac{1}{2\sqrt{N}}K_k H_{b,k}(S_k^{z,T} + S_k^z)H_{b,k}^T \\ &= \frac{1}{\sqrt{N}}K_k H_{b,k}S_k^z H_{b,k}^T \end{aligned} \quad (52)$$

Then

$$\begin{aligned} \frac{\partial J}{\partial K_k} = & 2(1 - \eta)(K_k H_{x,k} - I)C_k^- H_{x,k}^T + 2(1 - \eta)K_k H_{v,k}C_k^z H_{v,k}^T \\ & + 2\eta(1 + \frac{1}{\beta})(K_k H_{x,k} - I)S_k^- H_{x,k}^T + 2\eta(1 + \beta)K_k H_{b,k}S_k^z H_{b,k}^T. \end{aligned} \quad (53)$$

Let $\frac{\partial G_1}{\partial K_k} = 0$ and solve for K_k , we get an adaptive Kalman gain:

$$\begin{aligned} K_k = & [(1 - \eta)C_k^- H_{x,k}^T + \eta(1 + \frac{1}{\beta})S_k^- H_{x,k}^T] \cdot [(1 - \eta)H_{x,k}C_k^- H_{x,k}^T \\ & + (1 - \eta)H_{v,k}C_k^z H_{v,k}^T + \eta(1 + \frac{1}{\beta})H_{x,k}S_k^- H_{x,k}^T + \eta(1 + \beta)H_{b,k}S_k^z H_{b,k}^T]^{-1} \end{aligned} \quad (54)$$

Now we get the elicited expression of the cost function (46) by collecting (40), (45) and (54). All the procedures in this filtering step rely on the solution of the following optimization problem.

$$\begin{aligned} \min_{\beta} J(\beta) \\ \text{s.t. } \beta \in [0, +\infty) \subset \mathbb{R}^1 \end{aligned} \tag{55}$$

where the cost function $J(\beta)$ was defined in (46) and the solution of above optimization problem was denoted by β^* . Putting β^* into (54), (40) and (45) and we finished the filtering step.

Here are three remarks about this optimization problem:

- (1) Problem (55) is a nonlinear programming problem since the objective function (46) is nonlinear.
- (2) Problem (55) is a convex optimization problem [21]. Therefore, any existing local minimum is a global minimum.
- (3) Usually, it is hard to solve a nonlinear programming problem due to the constrained equations or inequalities. MATLAB function *fminsearch* is an efficient way to solve the problem (55). Further, an advanced toolbox *INTLAB* can also be used [22].

The parameter η was introduced to balance the random uncertainty and set-membership uncertainty. There are three very interesting cases need to be noticed [13].

When $\eta = \frac{1}{2}$, the stochastic uncertainty and set-membership uncertainty have the same weight and $K(p)$ contains no α in this case. This solution is

recommended to users when there is no expert-based information about η available.

When $\eta = 0$,

$$K_k(\beta) = C_k^- H_{x,k}^T \cdot [H_{x,k} C_k^- H_{x,k}^T + H_{v,k} C_k^z H_{v,k}^T]^{-1} \quad (56)$$

which is exactly the Kalman gain in the standard EKF [23].

When $\eta = 1$, the model now only contains set-membership uncertainty. In this case,

$$K_k(\beta) = (1 + \frac{1}{\beta}) S_k^- H_{x,k}^T \cdot [(1 + \frac{1}{\beta}) H_{x,k} S_k^- H_{x,k}^T + (1 + \beta) H_{b,k} S_k^z H_{b,k}^T]^{-1}. \quad (57)$$

4 Algorithm

An algorithm for SKF was summarized according to previous derivation.

Algorithm 1 Set-membership Kalman filter model

1: **Initialization:**

- (1) Initial state midpoint $\hat{x}_0^{c+} = x_0$.
- (2) Initial estimated random covariance matrix C_0^+ .
- (3) Initial estimated set-membership shape matrix S_0^+ .

2: **for** $k=1,2,\dots,K$ **do**

3: **Input of Prediction Step:**

- (1) Point post-estimation \hat{x}_k^+ , with estimated covariance C_k^+ and shape matrix S_k^+ .
- (2) Nonlinear system model

$$x_{k+1} = f_k(x_k, u_k, w_k, a_{1,k}, a_{2,k}, \dots, a_{I,k}), \quad (58)$$

where $w_k \sim \mathcal{N}(0, C_k^u)$ and $a_{i,k} \in \mathcal{E}(0, S_{i,k}^u)$, $i = 1, 2, \dots, I$.

- (3) Control input u_k , random noise covariance C_k^u and shape matrices $S_{i,k}^u$, $i = 1, 2, \dots, I$. for set-membership uncertainty.

4: **Calculation of Prediction Step:**

- (1) Computation of error covariance matrix C_{k+1}^- according to

$$C_{k+1}^- = F_{x,k} C_k^+ F_{x,k}^T + F_{w,k} C_k^u F_{w,k}^T. \quad (59)$$

- (2) The center of the priori ellipsoid:

$$\hat{x}_{k+1}^{c-} = F_{x,k} \hat{x}_k^{c+} + \tilde{u}_k. \quad (60)$$

- (3) The shape matrix of the priori ellipsoid:

$$S_{k+1}^- = \left(\sqrt{\text{tr}(F_{x,k} S_k^+ F_{x,k}^T)} + \sum_{i=1}^I \sqrt{\text{tr}(F_{a,i} S_{i,k}^u F_{a,i}^T)} \right) \cdot \left(\frac{F_{x,k} S_k^+ F_{x,k}^T}{\sqrt{\text{tr}(F_{x,k} S_k^+ F_{x,k}^T)}} + \sum_{i=1}^I \frac{F_{a,i} S_{i,k}^u F_{a,i}^T}{\sqrt{\text{tr}(F_{a,i} S_{i,k}^u F_{a,i}^T)}} \right). \quad (61)$$

The predicted point estimate \hat{x}_{k+1}^- is characterized by the random error

C_{k+1}^- and the set-membership error by \hat{x}_{k+1}^{c-} and S_{k+1}^- .

5: **Output of Prediction Step:**

Priori estimated state: \hat{x}_k^{c-} , C_k^- , and S_k^- .

6: **Input of Filtering Step:**

- (1) Priori or predicted estimate \hat{x}_k^- with error covariance matrix C_k^- and ellipsoid center \hat{x}_k^{c-} and shape matrix S_k^- .
- (2) Nonlinear measurement model:

$$y_k = h_k(x_k, v_k, b_k), \quad (62)$$

where $v_k \sim \mathcal{N}(0, C_k^z)$ and $b_k \in \mathcal{E}(0, S_k^z)$.

- (3) Observation y_k , sensor noise with random covariance C_k^z and set-membership shape matrix S_k^z .

(4) $\tilde{z}_k(\hat{x}_k^-) = h_k(\hat{x}_k^-, 0, 0) - H_{x,k}\hat{x}_k^-$.

- (5) Weighting parameter η .

7: **Calculation of Filtering Step:**

- (1) For given weighting parameter η , the optimal gain factor K_k is

$$\begin{aligned} K_k(\beta) = & [(1-\eta)C_k^- H_{x,k}^T + \eta(1 + \frac{1}{\beta})S_k^- H_{x,k}^T] \cdot [(1-\eta)H_{x,k}C_k^- H_{x,k}^T \\ & + (1-\eta)H_{v,k}C_k^z H_{v,k}^T + \eta(1 + \frac{1}{\beta})H_{x,k}S_k^- H_{x,k}^T \\ & + \eta(1 + \beta)H_{b,k}S_k^z H_{b,k}^T]^{-1}. \end{aligned} \quad (63)$$

- (2) Computation of the center of updated estimate \hat{x}_k^+ by means of

$$\hat{x}_k^{c+} = (I - K_k H_{x,k})\hat{x}_k^{c-} + K_k[y - \tilde{z}_k(\hat{x}_k^{c-})]. \quad (64)$$

- (3) Computation of updated error covariance matrix C_k^+ by

$$C_k^+(\beta) = (I - K_k H_{x,k})C_k^-(I - K_k H_{x,k})^T + K_k H_{v,k}C_k^z H_{v,k}^T K_k^T. \quad (65)$$

- (4) Update the shape matrix S_k^+ by

$$S_k^+(\beta) = (1 + \frac{1}{\beta})(I - K_k H_{x,k})S_k^-(I - K_k H_{x,k})^T + (1 + \beta)K_k H_{b,k}S_k^z H_{b,k}^T K_k^T. \quad (66)$$

- (5) The optimal parameter β^* can be solved by

$$\beta^* = \arg \min\{(1 - \eta)\text{tr}[C_k^+(\beta)] + \eta\text{tr}[S_k^+(\beta)]\}. \quad (67)$$

The updated point estimate \hat{x}_k^+ is characterized by random error characteristic $C_k^+ = C_k^+(\beta)$ and set-membership error description $S_k^+ = S_k^+(\beta)$. Put β^* into above 4 functions to get the optimal output.

8: **Output of Filtering Step:**

Posteriori estimated state: \hat{x}_k^{c+} , $C_k^+(\beta^*)$, and $S_k^+(\beta^*)$.

9: **end for**

5 Applications

5.1 Example 1: Highly Nonlinear Benchmark Example

Consider the following example:

$$x_{k+1} = \frac{1}{2}x_k + \frac{25x_k}{1+x_k^2} + 8\cos[1.2(k-1)] + w_k + a_k, \quad (68)$$

$$y_k = \frac{1}{20}x_k^2 + v_k + b_k. \quad (69)$$

where x_k is a scalar, $u_k = 8\cos[1.2(k-1)]$ is the input vector, $w_k \sim \mathcal{N}(0, 1)$ is a Gaussian process noise, $a_{i,k} \in \mathcal{E}(0, 9)$ is the unknown but bounded perturbation, in this 1-D case the ellipsoid is the interval $[-3, 3]$. $v_k \sim \mathcal{N}(0, 1)$ is the a Gaussian measurement noise, and $b_k \in \mathcal{E}(0, 4)$ is the unknown but bounded perturbation in the interval $[-2, 2]$. Initial true state is $x_0 = 0.1$, initial state estimate as $\hat{x}_0 = x_0$, initial estimation covariance matrix is $C_0^+ = 2$ and initial shape matrix is $S_0^+ = 1 \times 10^{-3}$. We used a simulation length of 50 time steps. Weight parameter $\eta = 0.5$.

This system was regarded as a benchmark in the nonlinear estimation theory [24][25], and it is usually used to demonstrate the drawbacks of EKF

comparing with particle filter [23]. The high degree of nonlinearity in both the process and measurement equations makes this system a very difficult state estimation problem for a Kalman filter. We use this example to show the new SKF behaves better than the traditional first order EKF when some set-membership uncertainties are included in the system.

We repeated this simulation for 100 times. And Fig. 2 shows the comparison results between SKF and EKF at time $k = 25, 50, 75, 100$.

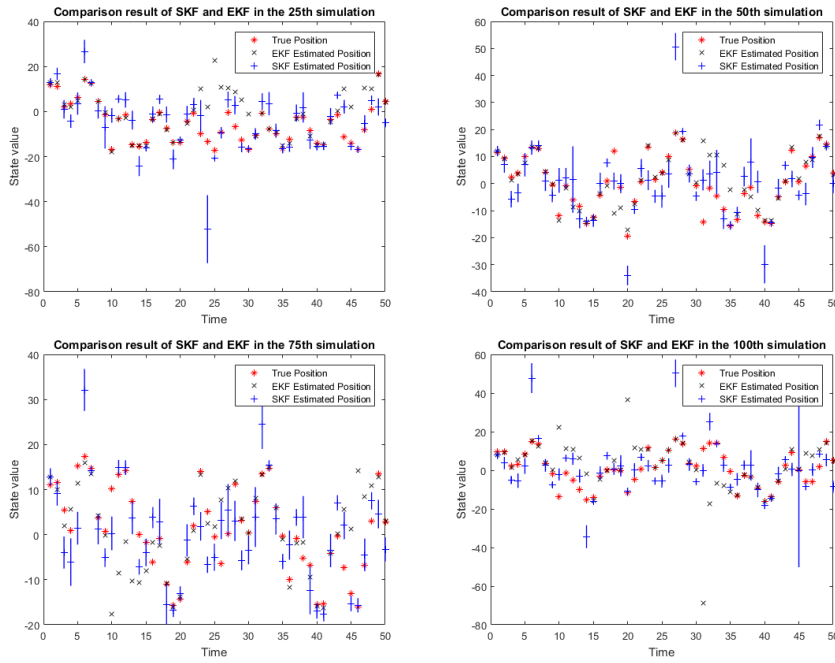


Fig. 2: Comparison results in 4 simulations

In above figures, the red stars denote the true positions of the state, the black crosses represent the estimated positions via EKF, the blue lines give the estimated ellipsoids (in this 1D case they are intervals) via SKF, and the blue

k	1	11	21	31	41	51	61	71	81	91
S	53.59	116.38	70.38	102.95	100.16	75.81	82.11	48.71	66.30	81.55
E	72.49	116.64	230.08	234.69	80.73	31.32	109.53	79.69	8.63	64.95

Table 1: Comparison of SKF and EKF in 10 simulations

plus signs mark the centers of the output ellipsoids. The center of the ellipsoid given by the new SKF is different with the traditional estimation via EKF, as what we expected, the ellipsoids include the true positions sometimes.

To further compare this new method with EKF, we calculate the distance vectors d_s, d_e of SKF and EKF with the true states, respectively. Each distance vector is 50×1 for the total 50 steps in every simulation. Table 1 shows the detailed l_2 norm comparison of these two distance vectors in 10 simulations ($k = 1, 11, \dots, 91$). The second row headed by S shows the distance error via SKF, and the third row headed by E shows the counterpart via EKF. We use the l_2 norm here as a generic measure of the distance between the estimated data and the true data, but other norms like l_1 and l_∞ are possible for use. Without loss of generality, we choose the midpoints of these ellipsoids for comparison.

In the whole 100-time simulation experiments, the overall l_2 norm of the distance vector under SKF is 148.70, with its counterpart in extended Kalman filter 192.29. The new SKF behaved much better than EKF in these 100 simulations. However, this does not mean the SKF is always a better algorithm comparing with EKF, since it is also possible to get opposite results when repeating this experiment.

5.2 Example 2: Two-Dimensional Trajectory Estimation

A vehicle moves on a plane with a curved trajectory [26]. The state vector $x = (x, y, v_x, v_y)$ contains positions and velocities of the target, in x-direction and y-direction, respectively. After linearization, we do not consider the acceleration process anymore, and the mathematical model of this vehicle was assumed as following:

$$\mathbf{x}_{k+1} = F_k \mathbf{x}_k + \mathbf{w}_k + \mathbf{a}_k \quad (70)$$

where $\mathbf{x}_k = (x_k, y_k, v_{x,k}, v_{y,k})$ is the state vector at time t_k . The transition matrix F_k is designed by:

$$F_k = \begin{pmatrix} 1 & 0 & dt & 0 \\ 0 & 1 & 0 & dt \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (71)$$

\mathbf{w}_k which representing random uncertainty is gaussian with covariance matrix C_k^u , and \mathbf{a}_k is the unknown but bounded uncertainty, which was bounded by an ellipsoid with shape matrix S_k^u . In total 300 points were observed and time step $dt = 0.1$ seconds. The units of time, distance, angle are second, meter and degree, respectively.

In this experiment, two observation stations $S_1 = [s_{12}, s_{12}]$ and $S_2 = [s_{21}, s_{22}]$ were arranged to make the measurements. Each station measured the distance and the direction angle of the vehicle. Here is the measurement equation:

$$\mathbf{y}_k = h_k(\mathbf{x}_k, v_k, b_k). \quad (72)$$

$$\mathbf{y}_k = \begin{pmatrix} d_1 \\ d_2 \\ \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} \sqrt{[x - s_{11}]^2 + [y - s_{12}]^2} \\ \sqrt{[x - s_{21}]^2 + [y - s_{22}]^2} \\ \arctan[(y - s_{12})/(x - s_{11})] \\ \arctan[(y - s_{22})/(x - s_{21})] \end{pmatrix} + \mathbf{v}_k + \mathbf{b}_k \quad (73)$$

\mathbf{v}_k which representing random uncertainty is gaussian with covariance matrix C_k^z , and \mathbf{b}_k is the unknown but bounded uncertainty, which was bounded by an ellipsoid with shape matrix S_k^z .

The initial state, estimated covariance matrix and shape matrix are given by: $x_0 = (0, 0, 0, 0)$, $C_0^+ = \text{diag}(0.01, 0.01, 0.01, 0.01)$ and $C_0^+ = \text{diag}(1 \times 10^{-6}, 1 \times 10^{-6}, 1 \times 10^{-6}, 1 \times 10^{-6})$.

The initial covariance matrices in process equation and measurement equation are given by:

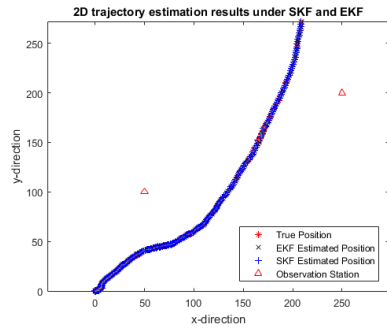
$$C_0^u = \begin{pmatrix} 0.0033 & 0 & 0.005 & 0 \\ 0 & 0.0033 & 0 & 0.005 \\ 0.005 & 0 & 0.01 & 0 \\ 0 & 0.005 & 0 & 0.01 \end{pmatrix}, C_0^z = \begin{pmatrix} 0.005^2 & 0 & 0 & 0 \\ 0 & 0.005^2 & 0 & 0 \\ 0 & 0 & 0.005^2 & 0 \\ 0 & 0 & 0 & 0.005^2 \end{pmatrix}. \quad (74)$$

The initial shape matrices of set-membership uncertainties in process equation and measurement equation are setting by:

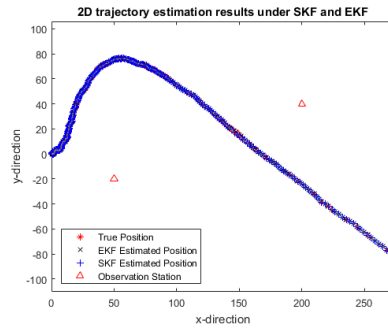
$$S_0^u = \begin{pmatrix} 1^2 & 0 & 0 & 0 \\ 0 & 1^2 & 0 & 0 \\ 0 & 0 & 0.5^2 & 0 \\ 0 & 0 & 0 & 0.5^2 \end{pmatrix}, S_0^z = \begin{pmatrix} 0.01^2 & 0 & 0 & 0 \\ 0 & 0.01^2 & 0 & 0 \\ 0 & 0 & (\frac{\pi}{180})^2 & 0 \\ 0 & 0 & 0 & (\frac{\pi}{180})^2 \end{pmatrix}. \quad (75)$$

Weight parameter $\eta = 0.5$.

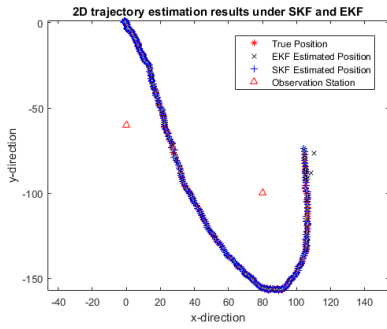
Below eight different trajectories were estimated by EKF and SKF from eight different data sets. The following Fig. 3 shows the estimation results. Red stars mark the true position according to the reference data, black crosses denotes the estimated position via EKF, and the blue plus signs are the geometry centers of the ellipsoids.



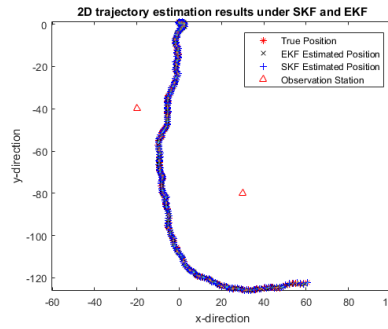
(a) Trajectory 1



(b) Trajectory 2



(c) Trajectory 3



(d) Trajectory 4

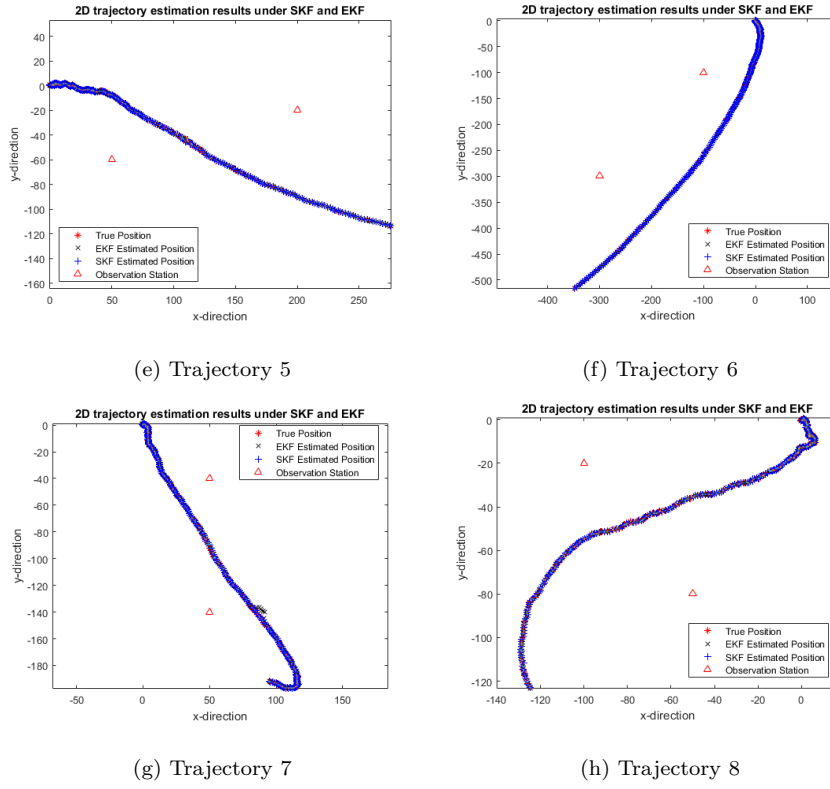


Fig. 3: Eight trajectories examples to compare SKF with EKF

One may notice that both EKF and SKF perform well in most part of each trajectory, except that the ellipsoids getting large in the interaction area between the trajectory and the straight line of two stations. Again, we calculate the l_2 norm of distance vectors to make further comparison in Table 2.

Trajectory	1	2	3	4	5	6	7	8
SKF	3.22	2.88	4.36	2.47	3.70	12.37	2.83	1.91
EKF	5.37	3.62	27.81	3.82	3.67	7.79	13.31	3.27

Table 2: Comparison of SKF and EKF in 8 trajectories

To check the estimation errors, we chose Trajectory 5 to repeat for 100 times and then get the following error distribution.

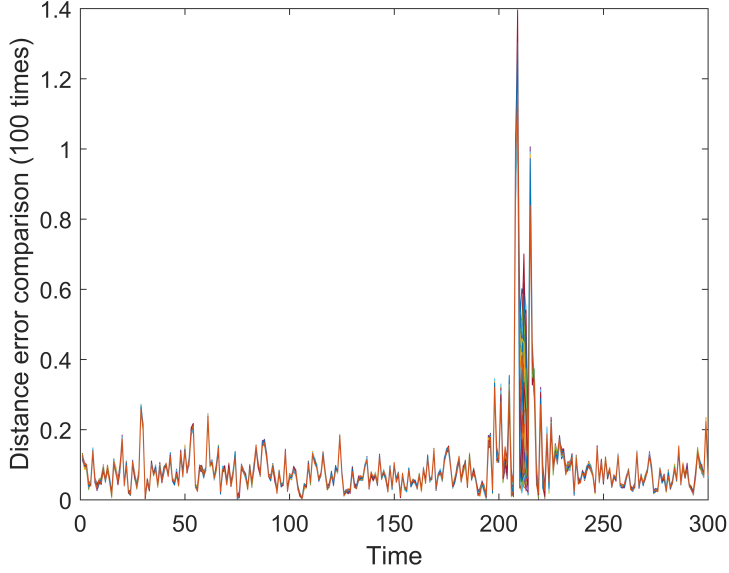


Fig. 4: Distance Error of SKF in 100 times

From above Fig. 4 it is obvious to notice that the estimated error was getting larger when $k \in [200, 225]$, i.e., in Fig. 3 (e) one may get worse estimation results in the intersection area of the line between the two observation stations and the trajectory of the vehicle. The following Fig. 5 shows more local details in the interaction area of Trajectory 5, where the straight line connects the two observation stations. Not only the estimated ellipsoids getting larger in the interaction area, but the semi-major axes of the largest ellipsoid is perpendicular to the straight line.

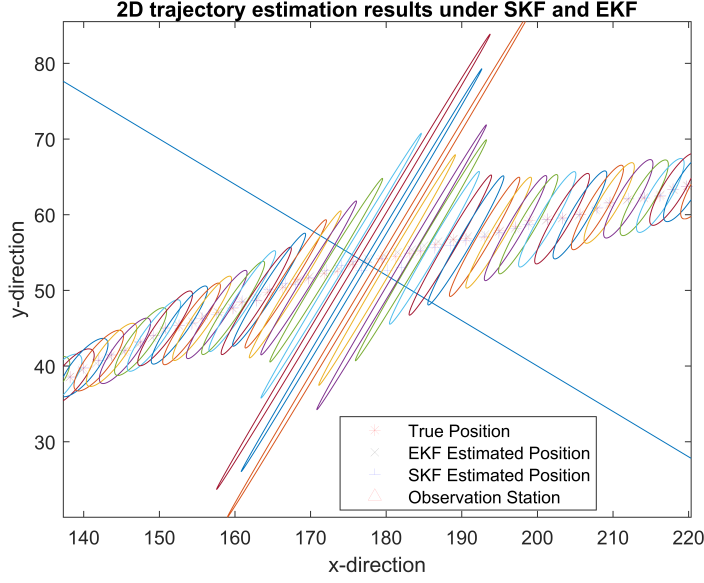


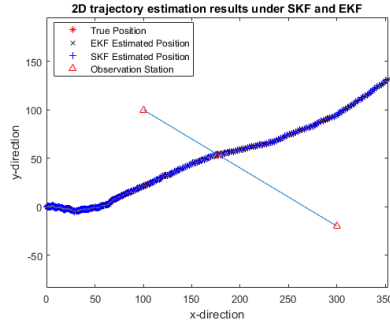
Fig. 5: Local estimation details near the interaction area

There exist two major reasons causing this phenomena.

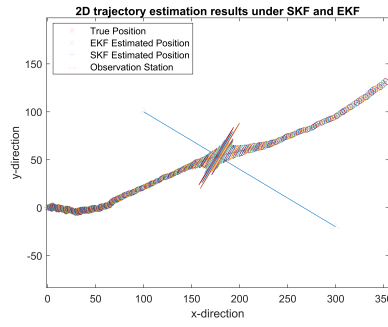
Firstly, the angle set-membership uncertainty played a more significant role in the estimation. From (75) we notice that in Fig. 3 Trajectory 5, the set-membership uncertainty of distance is $[-0.01, 0.01]$ meter, and its counterpart in angle is $[-1, 1]$ degree. The distance between the observation station and the interaction area is at least 85 meters, i.e., the uncertainty caused by angles would be $80 \times \frac{\pi}{180} \text{m} = 1.4835 \text{m}$ (in the vertical direction of the straight line), which is greatly larger than the distance uncertainties 0.01m (in the parallel direction of the straight line).

Secondly, the criterion of the optimization problem in (67) in the SKF algorithm is the trace of a shape matrix. There are several minimum criterions

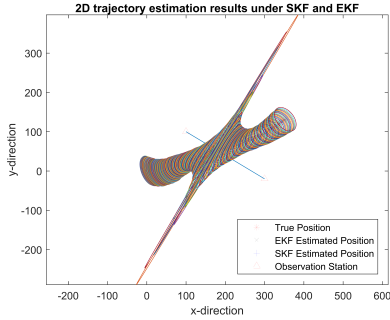
to get one optimal ellipsoid given a shape matrix S , e.g., the trace of the shape matrix $\text{tr}(S)$, the determinant of the shape matrix $|S|$, and the largest eigenvalue of the shape matrix $\lambda_M(S)$. Minimizing the largest eigenvalue $\lambda_M(S)$ smoothes the mean curvature and makes the ellipsoid more like a ball (circular in 2D case). Minimizing the trace or the determinant of the shape matrix produces an ellipsoid with small volume, but sometimes causes the ellipsoid getting oblate, i.e., more uncertainties in one certain direction in this example.



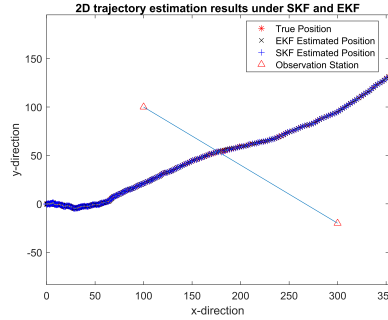
(a) Trajectory 5.1



(b) Trajectory 5.2



(c) Trajectory 5.3



(d) Trajectory 5.4

Fig. 6: Output ellipsoids are highly related to the initial settings.

Fig. 6 shows that the output estimated ellipsoids are highly related to the initial boundary of the set-membership uncertainties in both equations. The principle semi-axes in Trajectory 5.3 ($S_u = \text{diag}([100^2, 100^2, 50^2, 50^2])$, $S_z = \text{diag}([1^2, 1^2, (\pi/1.8)^2, (\pi/1.8)^2])$) is 100 times larger than their counterparts in Trajectory 5.1 ($S_u = \text{diag}([1^2, 1^2, 0.5^2, 0.5^2])$, $S_z = \text{diag}([0.01^2, 0.01^2, (\pi/180)^2, (\pi/180)^2])$) and 10 times larger than their counterparts in Trajectory 5.2 ($S_u = \text{diag}([10^2, 10^2, 5^2, 5^2])$, $S_z = \text{diag}([0.1^2, 0.1^2, (\pi/18)^2, (\pi/18)^2])$), the the outputs of the SKF are getting very large. Both the input (accuracy of the instruments) and the output (estimated positions) in Trajectory 5.2 and 5.3 are not realistic and one more realistic example was shown in Trajectory 5.4 with initial shape matrix $S_u = \text{diag}([0.5^2, 0.5^2, 0.5^2, 0.5^2])$, $S_z = \text{diag}([0.01^2, 0.01^2, (\pi/180)^2, (\pi/180)^2])$.

6 Conclusion and Future Work

One cannot state that the new SKF is always better than the standard EKF, however, the performance of SKF is much more reliable than EKF in some cases (like in previous simulated experiments). To say the least, the SKF is one reasonable and applicable model when some unknown but bounded uncertainties were included in the nonlinear system. A difference with the standard Kalman filter is that, the estimated states are ellipsoids instead of single points, and every inner points of one ellipsoid have the same estimation status. But one still can choose a series of particular points in these ellipsoids

if necessary. The output is reasonable considering the unknown but bounded uncertainties which were included in the original system, and extra information in the measurement equation was issued properly in the filtering step.

Like other filter models, there is also some space for this SKF to improve. For instance, the shape matrices of the set-membership uncertainties in both system and measurement equation must be given properly at the beginning, and also the weighting parameter should be decided by the user or experts.

The future work of our research includes deriving a similar algorithm for second order extend Kalman filter or unscented Kalman filter, using zonotopes or interval boxes to bound the unknown but bounded uncertainty, and minimizing the determinant or the largest eigenvalue of the shape matrix when solving the optimization problem. Last but not least, the stability of this algorithm should be carefully discussed considering that the state estimation problem is usually ill-posed as an inverse problem [27].

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