What Is the Economically Optimal Way to Guarantee Interval Bounds on Control?

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Abstract

For control under uncertainty, interval methods enable us to find a box $B = [\underline{u}_1, \overline{u}_1] \times \ldots [\underline{u}_n, \overline{u}_n]$ for which any control $u \in B$ has the desired properties – such as stability. Thus, in real-life control, we need to make sure that $u_i \in [\underline{u}_i, \overline{u}_i]$ for all parameters u_i describing control. In this paper, we describe the economically optimal way of guaranteeing these bounds.

Keywords: Interval uncertainty, Actuators, Control, Economically optimal solution

1 Formulation of the Problem

Control problems: a very brief reminder. In control problems, we need to find the values of the control

$$u = (u_1, \ldots, u_n).$$

Usually, there are some requirements on the control: e.g., that under this control, the system should be stable, etc. These conditions are usually described by inequalities; see, e.g., [1].

From optimal control to constraint satisfaction. In general, there are many different controls that satisfy all the desired constraints.

In the ideal case, when:

- we know the exact initial state of the system and
- we know the equations that describe the system's dynamics under different controls,

we can compute the exact consequences of each control. Thus, depending on what is our objective, we can select an appropriate objective functions and looks for the control that optimized this objective function.

The objective function depends on the task. For example, for selecting a plane trajectory, we can have different objective functions:

- In the situation of medical emergency, we need to find the trajectory of the plane that brings the medical team to the remote patient as soon as possible.
- For a regular passenger communications, we need to minimize expenses – and hence, instead o flying at the largest possible speed, we should fly at the speed that saves as much fuel as possible.
- For a private jet, a reasonable objective function is the ride's smoothness.

In practice, we rarely know the exact initial state and the exact system's dynamics. Often, for each of the corresponding parameters, we only know the lower and upper bounds on possible values, i.e., in other words, we only know the interval that contains all possible values of the corresponding parameter; see, e.g., [1–4]. In such cases, for each control, instead of the exact value of the objective function, we get the range of possible values $[\underline{v}, \overline{v}]$. Computing this range under interval uncertainty is a particular case of the main problem of *interval computations* [1–3]

In such situations, it make sense, e.g., to describe all the control u which are possibly optimal, i.e., for which

$$\overline{v}(u) \geq \max_{u'} \underline{v}(u').$$

What we get from interval computations. In situations of interval uncertainty, interval methods enable us to find a box

$$B = [\underline{u}_1, \overline{u}_1] \times \ldots [\underline{u}_n, \overline{u}_n]$$

for which any control $u \in B$ has the desired properties – such as stability or possible optimality.

Resulting problem. Thus, in real-life control, we need to make sure that

$$u_i \in [\underline{u}_i, \overline{u}_i]$$

for all parameters u_i describing control.

What is the most economical way to guarantee these bounds?

2 Towards Formulating the Problem in Precise Terms

Analysis of the problem. Actuators are never precise, so we can only set up the control value u_i with some accuracy a_i. Thus, if we aim for the midpoint

$$u_{mi} \stackrel{\text{def}}{=} \frac{\underline{u}_i + \overline{u}_i}{2},$$

we will get the actual value u_i within the interval

$$[u_{mi} - a_i, u_{mi} + a_i].$$

The only way to guarantee that the control value is indeed within the desired interval is to measure it. Measurement are also never absolutely precise. Let us assume that we use a measuring instrument with accuracy ε_i. This means that for each actual value u_i of the corresponding parameter, the measured value $\widetilde{u}_i$ is somewhere within the interval

$$[u_i - \varepsilon_i, u_i + \varepsilon_i].$$

Based on the measurements, the only thing we can conclude about the actual (unknown) value u_i is that it belongs to the interval

$$[\widetilde{u}_i - \varepsilon_i, \widetilde{u}_i + \varepsilon_i].$$

We want to make sure that all the values from this interval are within the desired interval

$$[\underline{u}_i, \overline{u}_i],$$

i.e., that

$$\underline{u}_i \le \widetilde{u}_i - \varepsilon_i \tag{1}$$

and

$$\widetilde{u}_i + \varepsilon_i \le \overline{u}_i. \tag{2}$$

These inequalities must hold for all possible values

$$\widetilde{u}_i \in [u_i - \varepsilon_i, u_i + \varepsilon_i].$$

For the inequality (1) to hold for all these values, it is sufficient to require that this inequality holds for the smallest possible value

$$\widetilde{u}_i = u_i - \varepsilon_i,$$

i.e., that we have

$$\underline{u}_i \le (u_i - \varepsilon_i) - \varepsilon_i = u_i - 2\varepsilon_i. \tag{3}$$

Similarly, for the inequality (2) to hold for all the values

$$\widetilde{u}_i \in [u_i - \varepsilon_i, u_i + \varepsilon_i],$$

it is sufficient to require that this inequality holds for the largest possible value

$$\widetilde{u}_i = u_i + \varepsilon_i,$$

i.e., that we have

$$(u_i + \varepsilon_i) + \varepsilon_i = u_i + 2\varepsilon_i \le \overline{u}_i. \tag{4}$$

The inequalities (3) and (4) must hold for all possible values $u_i \in [u_{mi} - a_i, u_{mi} + a_i]$. For the inequality (3) to hold for all these values, it is sufficient to require that this inequality holds for the smallest possible value $u_i = u_{mi} - a_i$, i.e., that we have

$$\underline{u}_i \le u_{mi} - 2\varepsilon_i - a_i. \tag{5}$$

Similarly, for the inequality (4) to hold for all the values $u_i \in [u_{mi} - a_i, u_{mi} + a_i]$, it is sufficient to require that this inequality holds for the largest possible value $u_i = u_{mi} + a_i$, i.e., that we have

$$u_{mi} + 2\varepsilon_i + a_i \le \overline{u}_i. \tag{6}$$

Let us denote the half-width of the interval

$$[\underline{u}_i, \overline{u}_i]$$

by

$$\Delta_i \stackrel{\text{def}}{=} \frac{\overline{u}_i - \underline{u}_i}{2}.$$

In terms of the half-width,

$$\overline{u}_i - u_{mi} = u_{mi} - \underline{u}_i = \Delta_i.$$

Thus, the inequalities (5) and (6) are equivalent to the inequality

$$2\varepsilon_i + a_i \le \Delta_i. \tag{7}$$

In the optimal solution, we have equality. In general, the more accuracy we want, the more expensive will be the corresponding measurements and actuators. From this viewpoint, if

$$2\varepsilon_i + a_i < \Delta_i,$$

then we can use slightly less accurate actuators and/or measuring instruments and still guarantee the desired inequality (7). Thus, in the most economical solution, in the formula (7), we should have the exact equality:

$$2\varepsilon_i + a_i = \Delta_i, \tag{8}$$

i.e., equivalently,

$$a_i = \Delta_i - 2\varepsilon_i. \tag{9}$$

Towards resulting formulation of the problem. So, the problem is to find. among all the values a_i and ε_i that satisfy the equality (9), the values for which the overall expenses are the smallest possible.

To solve this problem, we need to know how the cost of actuators and measurements depends on accuracy. To analyze this dependence, we start with a 1-D case, when we only have a single control parameter u_1.

3 1-D Case, When We Have a Single Control Parameter u_1

Cost of actuators: main idea. Actuators – such as robotic arms – are usually rather crude, so we may not be able to properly orient the robot after the first attempt. A natural way to provide a better accuracy is to repeat the attempts until we get the desired location (or, in general, the desired value of the parameter u_i).

Cost of actuators: from idea to a formula. Let us assume that the given actuator can provide the value u_i with some accuracy A_i. This means that if we aim for the midpoint u_{mi}, we will get values from the interval

$$[u_{mi} - A_i, u_{mi} + A_i].$$

We do not know the relative frequency of different values within this interval. Since we have no reasons to assume that some of these values are more probable and some are less probable, it is therefore reasonable to assume that all the values from the interval

$$[u_{mi} - A_i, u_{mi} + A_i]$$

are equally probable, i.e., that we have a uniform probability distribution on this interval.

By repeatedly trying, we want to get the value u_i within the interval

$$[u_{mi} - a_i, u_{mi} + a_i]$$

for some

$$a_i < A_i.$$

For the uniform distribution, the probability to be within a subinterval is proportional to the width of this subinterval – namely, it is equal to the ratio between the width of the subinterval and the width of the entire interval. In particular, the probability that at each try, the value u_i is within the interval is equal to the ratio

$$p = \frac{2a_i}{2A_i} = \frac{a_i}{A_i}.$$

So, on average, we need

$$\frac{1}{p} = \frac{A_i}{a_i}$$

iterations to get into the desired interval

$$[u_{mi} - a_i, u_{mi} + a_i].$$

Cost of measurements. How to gauge the cost of accurate measurements? It is known that if we start with a measuring instrument with 0 mean and standard deviation σ_i, then, by performing M independent measurements and averaging the results, we get a $\sqrt{M}$ times smaller standard deviation.

Indeed, the variance of the sum of independent random variables is equal to the sum of the variances, so for the sum of M measurement errors, the variance is $m \cdot \sigma_i^2$, and thus, the standard deviation is $\sqrt{m} \cdot \sigma_i$. The arithmetic average is obtained by dividing the sum by M, so its standard deviation is

$$\frac{\sqrt{M} \cdot \sigma_i}{M} = \frac{\sigma_i}{\sqrt{M}}.$$

So, if we start with a measuring instrument with accuracy σ_i, and we want accuracy ε_i, we need to repeat each measurement M_i times, where

$$\frac{\sigma_i}{\sqrt{M_i}} = \varepsilon_i,$$

i.e.,

$$M_i = \frac{\sigma_i^2}{\varepsilon_i^2}.$$

The cost of measurement is proportional to the number M_i of such measurements, so it is equal to

$$m_i \cdot M_i = m_i \cdot \frac{\sigma_i^2}{\varepsilon_i^2},$$

where by m_i, we denoted the cost of a single measurement.

Overall cost. Let T_i denote the cost of a single actuator try.

The overall cost of one try of an actuator is equal to the actuator trying cost T_i plus the measurement cost $m_i \cdot \frac{\sigma_i^2}{\varepsilon_i^2}$, i.e., it is equal to

$$T_i + m_i \cdot \frac{\sigma_i^2}{\varepsilon_i^2}.$$

To achieve the actuator accuracy

$$a_i = \Delta_i - 2\varepsilon_i,$$

we need to perform

$$\frac{A_i}{a_i} = \frac{A_i}{\Delta_i - 2\varepsilon_i}$$

tries. Thus, the overall cost C of all the tries is equal to

$$C = \frac{A_i}{\Delta_i - 2\varepsilon_i} \cdot \left(T_i + m_i \cdot \frac{\sigma_i^2}{\varepsilon_i^2} \right). \tag{10}$$

Resulting optimization problem. In the 1-D case, we need to find the value ε_i for which the cost (10) is the smallest possible.

Discussion.

- When ε_i is close to 0, the cost of measurement tends to infinity, so we have a very large overall cost.
- Similarly, when a_i is close to 0, i.e., when

$$\varepsilon_i \approx \frac{\Delta_i}{2},$$

the actuator cost becomes very large, so the overall cost is also very large.

Thus, there should be values ε_i between 0 and

$$\frac{\Delta_i}{2}$$

for which the cost of maintaining u_i within the desired interval

$$[\underline{u}_i, \overline{u}_i]$$

is the smallest possible.

How to solve this optimization problem. In the 1-D case, where we have a single unknown ε_i, to find the optimal value of this unknown, we can simply differentiate the objective function (1) with respect to ε_i and equate the derivative to 0.

Minimizing the expression (10) is equivalent to minimizing its logarithm

$$\ln(A_i) - \ln(\Delta_i - 2\varepsilon_i) + \ln\left(T_i + m_i \cdot \frac{\sigma_i^2}{\varepsilon_i^2}\right).$$

Differentiating this expression with respect to ε_i and equating the derivative to 0, we get

$$\frac{2}{\Delta_i - 2\varepsilon_i} - 2 \cdot m_i \cdot \sigma_i^2 \cdot \frac{1}{\varepsilon_i^3} \cdot \frac{1}{T_i + m_i \cdot \sigma_i^2 \cdot \varepsilon_i^{-2}} = 0.$$

Dividing both sides by 2, moving the negative term to the right-hand side, and explicitly multiplying the expressions in the right-hand side, we get

$$\frac{1}{\Delta_i - 2\varepsilon_i} = \frac{m_i \cdot \sigma_i^2}{\varepsilon_i^3 \cdot T_i + m_i \cdot \sigma_i^2 \cdot \varepsilon_i}.$$

Bringing both fractions to the common denominator, we get a cubic equation

$$\varepsilon_i^3 \cdot T_i + m_i \cdot \sigma_i^2 \cdot \varepsilon_i = m_i \cdot \sigma_i^2 \cdot (\Delta_i - 2\varepsilon_i),$$

i.e., equivalently,

$$T_i \cdot \varepsilon_i^3 + 3m_i \cdot \sigma_i^2 \cdot \varepsilon_i - m_i \cdot \sigma_i^2 \cdot \Delta_i = 0. \quad (11)$$

Resulting algorithm. To find the optimal accuracy ε_i of the measuring instrument, we can use one of the standard methods (e.g., Newton's method) so solve the cubic equation (11).

Then, we can find the optimal value a_i of the actuator accuracy as

$$a_i = \Delta_i - 2\varepsilon_i.$$

4 General Case

Notations. In general, we may have several actuators. Let us denote the number of actuators by A. For each actuator $a = 1, \ldots, A$, let us denote:

- the cost of one try by T_a, and
- the number of the corresponding control parameters by n_a.

For each actuator a, and for each i from 1 to n_a, let us denote:

- the i-th control parameter by u_{ai};

- the bounds of the corresponding control parameter by $\underline{u}_{ai}$ and $\overline{u}_{ai}$;
- the midpoint of the resulting interval by

$$u_{mai} \stackrel{\text{def}}{=} \frac{\underline{u}_{ai} + \overline{u}_{ai}}{2};$$

- the half-width of the corresponding interval by

$$\Delta_{ai} \stackrel{\text{def}}{=} \frac{\overline{u}_{ai} - \underline{u}_{ai}}{2};$$

- the bounds achievable on one try by A_{ai},
- the desired actuator accuracy by a_{ai},
- the desired measurement accuracy by ε_{ai},
- the accuracy of the corresponding measuring instrument by σ_{ai}, and
- the cost of a single measurement with that accuracy by m_{ai}.

Relation between accuracies of actuator and measurement. Similarly to the 1-D case, we can conclude that in the general case, for each a and i, we have

$$2\varepsilon_{ai} + a_{ai} = \Delta_{ai},$$

i.e.:

$$a_{ai} = \Delta_{ai} - 2\varepsilon_{ai}.$$

Cost of actuators. We assume that the actuator a can provide the value u_{ai} with some accuracy A_{ai}. This means that if we aim for the midpoint u_{mai}, we will get values from the interval

$$[u_{mai} - A_{ai}, u_{mai} + A_{ai}].$$

Similar to the 1-D case, it is reasonable to assume that all the combinations

$$(u_{a1}, \ldots, a_{an_a})$$

from the corresponding box

$$[u_{ma1} - A_{a1}, u_{ma1} + A_{a1}] \times \ldots \times [u_{man_a} - A_{an_a}, u_{man_a} + A_{an_a}]$$

are equally probable, i.e., that we have a uniform probability distribution on this box.

By repeatedly trying, we want to get the value

$$(u_{a1}, \ldots, u_{an_a})$$

within the smaller box

$$[u_{ma1} - a_{a1}, u_{ma1} + a_{a1}] \times \ldots \times [u_{man_a} - a_{an_a}, u_{man_a} + a_{an_a}].$$

For the uniform distribution, the probability to be within a sub-box is proportional to the volume of this sub-box – namely, it is equal to the ratio between the volume of the sub-box and the volume of the entire box. In particular, the probability that at each try, the values a_{ai} are within the desired box is equal to the ratio

$$p = \frac{(2a_1) \cdot \ldots \cdot (2a_{n_a})}{(2A_1) \cdot \ldots \cdot (2A_{n_a})} = \frac{a_{a1} \cdot \ldots \cdot a_{an_a}}{A_{a1} \cdot \ldots \cdot A_{an_a}}.$$

So, on average, we need

$$\frac{1}{p} = \frac{A_{a1} \cdot \ldots \cdot A_{an_a}}{a_{a1} \cdot \ldots \cdot a_{an_a}} = \prod_{i=1}^{n_a} \frac{A_{ai}}{a_{ai}}$$

iterations to get into the desired box

$$[u_{ma1} - a_{a1}, u_{ma1} + a_{a1}] \times \ldots \times [u_{man_a} - a_{an_a}, u_{man_a} + a_{an_a}].$$

Overall cost for each actuator. Similarly to the 1-D case, for each i, the cost of measuring the value u_{ai} with accuracy ε_{ai} is equal to

$$m_{ai} \cdot \frac{\sigma_{ai}^2}{\varepsilon_{ai}^2}.$$

Thus, the overall cost of measuring all these values is equal to the sum

$$\sum_{i=1}^{n_a} m_{ai} \cdot \frac{\sigma_{ai}^2}{\varepsilon_{ai}^2}.$$

The overall cost of one try of an actuator is equal to the actuator trying cost T_a plus the measurement cost:

$$T_a + \sum_{i=1}^{n_a} m_{ai} \cdot \frac{\sigma_{ai}^2}{\varepsilon_{ai}^2}.$$

To achieve the actuator accuracy

$$a_{ai} = \Delta_{ai} - 2\varepsilon_{ai},$$

we need to perform

$$\prod_{i=1}^{n_a} \frac{A_{ai}}{a_{ai}} = \prod_{i=1}^{n_a} \frac{A_{ai}}{\Delta_{ai} - 2\varepsilon_{ai}}$$

tries. Thus, the overall cost C_a of all the tries is equal to

$$C_a = \prod_{i=1}^{n_a} \frac{A_{ai}}{\Delta_{ai} - 2\varepsilon_{ai}} \cdot \left(T_a + \sum_{i=1}^{n_a} m_{ai} \cdot \frac{\sigma_{ai}^2}{\varepsilon_{ai}^2} \right). \tag{12}$$

The overall cost of all the actuators. The overall cost C of all the actuators can be obtained by adding up all the costs of all the actuators:

$$C = \sum_{a=1}^{A} C_a = \sum_{a=1}^{A} \left(\prod_{i=1}^{n_a} \frac{A_{ai}}{\Delta_{ai} - 2\varepsilon_{ai}} \cdot \left(T_a + \sum_{i=1}^{n_a} m_{ai} \cdot \frac{\sigma_{ai}^2}{\varepsilon_{ai}^2} \right) \right). \tag{13}$$

Resulting optimization problem. In the general case, we need to find the values ε_{ai} for which the cost (13) is the smallest possible.

Towards solving the optimization problem. First, one can notice that each cost C_a depends only on the parameters corresponding to this actuator. Thus, to optimize the overall cost C, it is sufficient to optimize the cost C_a for each actuator a.

For each a, minimizing C_a is equivalent to minimizing its logarithm

$$\sum_{i=1}^{n_a} \ln(A_{ai}) - \sum_{i=1}^{n_a} \ln(\Delta_{ai} - 2\varepsilon_{ai}) + \ln \left(T_a + \sum_{i=1}^{n_a} m_{ai} \cdot \frac{\sigma_{ai}^2}{\varepsilon_{ai}^2} \right).$$

Differentiating this expression with respect to ε_{ai} and equating the derivative to 0, we get

$$\frac{2}{\Delta_{ai} - 2\varepsilon_{ai}} - 2 \cdot m_{ai} \cdot \sigma_{ai}^2 \cdot \frac{1}{\varepsilon_{ai}^3} \cdot \frac{1}{m_a} = 0, \tag{14}$$

where we denoted

$$m_a \stackrel{\text{def}}{=} T_a + \sum_{j=1}^{n_a} m_{aj} \cdot \frac{\sigma_{aj}^2}{\varepsilon_{aj}^2}. \tag{15}$$

Dividing both sides of the formula (14) by 2, moving the negative term to the right-hand side, and explicitly multiplying the expressions in the right-hand side, we get

$$\frac{1}{\Delta_{ai} - 2\varepsilon_{ai}} = \frac{m_{ai} \cdot \sigma_{ai}^2}{\varepsilon_{ai}^3 \cdot m_a}.$$

Bringing both fractions to the common denominator, we get a cubic equation

$$\varepsilon_i^3 \cdot m_a = m_{ai} \cdot \sigma_{ai}^2 \cdot (\Delta_{ai} - 2\varepsilon_{ai}),$$

i.e., equivalently,

$$m_a \cdot \varepsilon_{ai}^3 + 2m_{ai} \cdot \sigma_{ai}^2 \cdot \varepsilon_{ai} - m_{ai} \cdot \sigma_{ai}^2 \cdot \Delta_{ai} = 0. \tag{16}$$

Resulting algorithm. For each actuator a, once we fixed the value m_a, we can find each value ε_{ai} $(i = 1, \ldots, n_a)$ by solving the cubic equation (16).

We can then check whether our guess was correct by checking whether the formula (15) is satisfied for the resulting values ε_{ai}. By using bisection, we can find the value m_a for which the equality (15) is satisfied.

Then, we can find the optimal value a_{ai} of the actuator accuracy as

$$a_{ai} = \Delta_{ai} - 2\varepsilon_{ai}.$$

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References

[1] L. Jaulin, M. Kiefer, O. Didrit, and E. Walter, *Applied Interval Analysis, with Examples in Parameter and State Estimation, Robust Control, and Robotics* (Springer, London, 2001).

[2] G. Mayer, *Interval Analysis and Automatic Result Verification* (de Gruyter, Berlin, 2017).

[3] R. E. Moore, R. B. Kearfott, and M. J. Cloud, *Introduction to Interval Analysis* (SIAM, Philadelphia, 2009).

[4] S. G. Rabinovich, *Measurement Errors and Uncertainty: Theory and Practice* (Springer Verlag, Berlin, 2005).

[5] A. Vaccaro and C. A.Cañizares, An affine arithmetic-based framework for uncertain power flow and optimal power flow studies, *IEEE Transactions on Powe Systems*, **32**(1), 274–288 (2017).