Use of Symmetries in Economics: An Overview

Vladik Kreinovich[1], Olga Kosheleva[1],
Nguyen Ngoc Thach[2], and Nguyen Duc Trung[2]
[1]University of Texas at El Paso
El Paso, Texas 79968, USA, vladik@utep.edu, olgak@utep.edu
[2]Banking University of Ho Chi Minh City
56 Hoang Dieu 2, Quan Thu Duc, Thu Duc
Ho Chi Minh City, Vietnam, Thachnn@buh.edu.vn

Abstract

In this paper, we show that many semi-heuristic econometric formulas can be derived from the natural symmetry requirements. The list of such formulas includes many famous formulas provided by Nobel-prize winners, such as Hurwicz optimism-pessimism criterion for decision making under uncertainty, McFadden's formula for probabilistic decision making, Nash's formula for bargaining solution – as well as Cobb-Douglas formula for production, gravity model for trade, etc.

1 Why Symmetries

How do people make predictions? How do people make predictions? How did people know that the Sun will rise in the morning? that a poisonous snake can bite, and its bite can be deadly? Because in the past, the sun was always rising; because in the past, snakes would sometimes bite, and the bitten person would sometimes die.

In all these cases, to make a prediction, we look at similar situations in the past – and make predictions based on what happened in such situations.

Some predictions are more complicated than that – they are based on using formulas, equations, and physical laws. But how do we know that a formula – e.g., Ohm's law – is valid? Because in several previous similar situations, this formula was true, so we conclude that this formula should be true now as well.

How to describe this idea in precise terms? The fact that the same phenomenon is observed in several similar situations means, in effect, that we can make some changes in a situation, and the conclusion will remain the same.

For example, when we check Ohm's law, we can move the laboratory – in which we perform the measurements – to a different location, we can rotate it, we can increase it in size, we can change the value of the current, and after all these changes, the formula remains the same – in other words, remains *invariant.*

Let us describe this invariance in precise terms. We have some phenomenon p depending on the situation s. A generic change – such as shift or rotations – means that we replace the original situation s by the changed situation $T(s)$. In these terms, invariance means that the phenomenon remains the same after the change, i.e., that

$$p(T(s)) = p(s). \quad (1)$$

In physics, such invariance is called a symmetry. A particular case of an invariance is when we have, e.g., a spherically symmetric object. If we rotate this object, it will remain the same – this is exactly what symmetry means in geometry.

Because of this example, physicists call each invariance *symmetry*.

Symmetries play a fundamental role in physics. Our above argument seems to indicate that symmetries play a fundamental role in physics – and indeed they do; see, e.g., [10, 42].

While on the past, new physical theories – such as Newton's mechanics or Maxwell's electromagnetism – were formulated in terms of differential equations, nowadays theories are usually formulated in terms of their symmetries, and equations can be derived from the requirement of invariance with respect to these symmetries. Moreover, it turned out that even more traditional physical equations, such as Newton's or Maxwell's, equations that were *not* originally derived from symmetries, can actually be uniquely determined by the corresponding symmetries; see, e.g., [11, 12, 22, 24].

Comment. Similar symmetries can be used to explain many algorithms and heuristics in computer science [37], including several heuristic formulas from fuzzy logic, the empirical efficiency of different activation functions in neural networks, etc.

What about economics? The above arguments about predictions are not limited to physical world: we make predictions about social events – e.g., economic predictions – the same way we make predictions in physics: we recall similar situations in the past, and we predict that the same phenomenon will occur now. In other words, predictions in economics are also, in essence, based on invariance and symmetries.

So, the following natural question appears. As we have mentioned, in physics, many empirical formulas, formulas that were originally derived based on the observations, can often be derived from the basic symmetries. Can we do the same with empirical-based econometric formulas? Can we derive them from some basic symmetries?

Our answer to this question. Our answer to the above questions is "Yes, we can!". In this paper, we will show that many basic semi-heuristic economic laws can actually be derived from the corresponding natural symmetries.

To explain how the economics laws can be thus derived, we first need to analyze which symmetries are natural in the economic context. In this analysis, we will follow an analogy with physics.

2 Which Symmetries Are Natural

Scaling: case of physics. Equations – like Ohm's law stating that the voltage V is equal to the product of the current I and the resistance R – deal with numerical values of different physical quantities. But these numerical values are not absolute, they depend on the choice of the measuring unit.

For example, if instead of using Ampere (A) as a unit of current we use a 1000 times smaller unit milli-Ampere (mA), the actual current will not change, but its numerical value will multiply by 1000. For example, instead of 2 A, we will now have $1000 \cdot 2 = 2000$ mA.

In general, if we replace the original measuring unit with a unit which is λ times smaller, then all the numerical values get multiplied by λ: instead of the original value x, we now have a new value $x' = \lambda \cdot x$. Such a transformation $x \to \lambda \cdot x$ that multiplies each value x by the same constant λ is known as *scaling*, and invariance with respect to scaling is known as *scale-invariance.*

What can we deduce from scale-invariance. Let us first consider the simplest case when we have a dependence of one quantity on the other $y = f(x)$. This is the case, e.g., if we fix a conductor (and thus, fix its resistance), and we analyze how the voltage y measured between the two ends of this conductor depends on the current x.

At first glance, it may seem that invariance simply means that when we replace x and $\lambda \cdot x$, the value of y should not change:

$$f(\lambda \cdot x) = f(x). \tag{2}$$

However, such a definition would lead to a constant function $f(x)$ (at least a function which is constant for $x > 0$): indeed, for every $q > 0$, by taking $x = 1$ and $\lambda = q$, we conclude, from the formula (2), that $f(q) = f(1)$, i.e., that the function $f(x)$ is indeed a constant.

From the physical viewpoint, the reason for this strange result is clear: different measuring units are related. For example, if we change a unit of distance from meters to feet, then, to preserve physical formulas, we also need to change the unit of speed from m/sec to ft/sec. Similarly, if we change the unit of current, then, to preserve the formulas, we need to appropriately change the unit for voltage. In general:

- if we change the unit of x to a λ times smaller one and thus change x to $x' = \lambda \cdot x$,
- then we should according change the unit of y to a one which is C times different: $y' = C \cdot y$, where this C depends on λ: $C = C(\lambda)$,
- so that when $y = f(x)$, then in the new units x' and y', we have the exact same dependence $y' = f(x')$.

Substituting the above expressions for x' and y' into the formula $y' = f(x')$, we conclude that

$$f(\lambda \cdot x) = C(\lambda) \cdot f(x). \tag{3}$$

What can we deduce from this scale-invariance? For simplicity, let us assume that the function $f(x)$ is differentiable – this is a usual assumption in physics. In this case, the function $C(\lambda) = \dfrac{f(\lambda \cdot x)}{f(x)}$ is also differentiable – as a ratio of two differentiable functions. Thus, we can differentiate both side of equation (3) with respect to λ and substitute $\lambda = 1$. As a result, we first get $x \cdot \dfrac{df}{dx}(\lambda \cdot x) = \dfrac{dC}{d\lambda}(\lambda) \cdot f(x)$, and then

$$x \cdot \frac{df}{dx}(x) = c \cdot f(x),$$

where we denoted $c \stackrel{\text{def}}{=} C'(1)$. We can now separate the variables, i.e., move all the terms containing x and dx to one side, and all the terms containing f and df to another side. For that, we multiply both sides by dx and divide both sides by x and f, getting $\dfrac{df}{f} = c \cdot \dfrac{dx}{x}$. Integrating both sides, we get $\ln(f) = c \cdot \ln(x) + c_0$, where c_0 is an integration constant. Thus,

$$f = \exp(\ln(f)) = \exp(c \cdot \ln(x) + c_0) = \exp(c \cdot \ln(x)) \cdot \exp(c_0) =$$

$$A \cdot (\exp(\ln(x))^c = A \cdot x^c,$$

where we denoted $A \stackrel{\text{def}}{=} \exp(c_0)$.

So, scale-invariance implies the power law $y = A \cdot x^c$.

Comments.

- This result holds without assuming that the function $f(x)$ is differentiable: it is sufficient to assume that it is continuous (or even measurable); see, e.g., [1].
- A similar result holds if we have a dependence on several variables, i.e., if we have a dependence $y = f(x_1, \ldots, x_n)$ which is scale-invariant in the sense that for each values $\lambda_1, \ldots, \lambda_n$, there exists a C such that if $y = f(x_1, \ldots, x_n)$ then $y' = f(x'_1, \ldots, x'_n)$, where $x'_i = \lambda_i \cdot x_i$ and $y' = C \cdot y$. Such functions have the form $y = A \cdot x_1^{c_1} \cdot \ldots \cdot x_n^{c_n}$.

Scale-invariance is important in economics as well. Many quantities in economics are scale-invariant: for example, the numerical values of income or of the country's Gross Domestic Product (GDP) depends on what monetary units we use. We can use the units of the corresponding country – e.g., Dong in the case of Vietnam – or, if we want to compare salaries in different countries, we can use one of the universal currencies, e.g., US dollars.

The actual income in the same no matter what units we use, but numerical values are, of course, different. Similar to physics, in such cases, it makes sense to require that the resulting formulas remain valid if we simply change a monetary unit; of course, we may need to appropriately change related units as well.

Shift: case of physics. For some physical quantities, the numerical value also depends on the starting point. For example, while we usually measure time by using Year 0 as the starting point, many religious calendars – corresponding to Buddhism, Islam, Judaism, etc. – use different starting times.

Similarly, while the usual Celsius scale for temperature starts with the water freezing point as 0, we an alternatively use the Kelvin scale, in which 0 is the smallest possible temperature ≈ -273 C, or the Fahrenheit scale commonly used in the US, in which 0 C corresponds to 32 F.

In general, if we replace the original starting point with a starting point which is x_0 times smaller or earlier, then all the numerical values are increase by x_0: instead of the original value x, we now have a new value $x' = x + x_0$. Such a transformation $x \to x + x_0$, that adds the same constant x_0 to each value x, is known as *shift*, and invariance with respect to shift is known as *shift-invariance.*

What can we deduce from shift-invariance. Let us first consider the case when we have a dependence of one quantity on the other $y = f(x)$. In this case, if we change the starting point for x, then, to preserve the formulas, we need to appropriately change the unit for y:

- if we change x to $x' = x + x_0$,
- then we should according change the unit of y to a one which is C times different $y' = C \cdot y$, where this C depends on x_0: $C = C(x_0)$,
- so that when $y = f(x)$, then in the new units x' and y', we have the exact same dependence $y' = f(x')$.

Substituting the above expressions for x' and y' into the formula $y' = f(x')$, we conclude that

$$f(x + x_0) = C(x_0) \cdot f(x). \tag{4}$$

What can we deduce from this shift-invariance? Let us assume that the function $f(x)$ is differentiable. In this case, the function $C(x_0) = \dfrac{f(x + x_0)}{f(x)}$ is also differentiable – as a ratio of two differentiable functions. Thus, we can differentiate both side of equation (4) with respect to x_0 and substitute $x_0 = 0$. As a result, we first get $\dfrac{df}{dx}(x + x_0) = \dfrac{dC}{dx_0}(x_0) \cdot f(x)$, and then

$$\frac{df}{dx} = c \cdot f,$$

where we denoted $c \stackrel{\text{def}}{=} C'(0)$.

We can now separate the variables, i.e., move all the terms containing x and dx to one side, and all the terms containing f and df to another side. For that, we multiply both sides by dx and divide both sides by f, getting $\dfrac{df}{f} = c \cdot dx$.

Integrating both sides, we get $\ln(f) = c \cdot x + c_0$, where c_0 is an integration constant. Thus,

$$f = \exp(\ln(f)) = \exp(c \cdot x + c_0) = A \cdot \exp(c \cdot x),$$

where we denoted $A \stackrel{\text{def}}{=} \exp(c_0)$.

So, shift-invariance implies the exponential dependence $y = A \cdot \exp(c \cdot x)$.

Comments.

- This result holds without assuming that the function $f(x)$ is differentiable: it is sufficient to assume that it is continuous (or measurable); see, e.g., [1].

- A similar result holds if we have a dependence on several variables, i.e., if we have a dependence $y = f(x_1, \ldots, x_n)$ which is shift-invariant in the sense that for each values $x_{01}, \ldots, x_{0n}$, there exists a C such that if $y = f(x_1, \ldots, x_n)$ then $y' = f(x'_1, \ldots, x'_n)$, where $x'_i = x_i + x_{0i}$ and $y' = C \cdot y$. Such functions have the form $y = A \cdot \exp(c_1 \cdot x_1 + \ldots + c_n \cdot x_n)$.

Shift-invariance is important in economics as well. Many quantities in economics are shift-invariant. For example, when we compute the income of people living in countries with socialized medicine, we can compute this income in two ways:

- we can simply take the income as is,

- or, if want a fair comparison with income in countries like US, where there is no socialized medicine, we add the average cost of medical expenses to the income.

Additivity. How can we estimate the force $f(q)$ with which an electric field acts on a body of a known electric charge q? If this body consists of two components, then there are two ways to do it:

- we can apply the formula $f(q)$ to the body as a whole,

- or we can apply this formula to both components, with charges q' and q'', find the forces $f' = f(q')$ and $f'' = f(q'')$ acting on each of the components, and then add these forces into a single value $f(q') + f(q'')$.

The second possibility come from the fact that both charges and forces are *additive* in the sense that:

- the overall electric charge q of a two-component body in which two components have electric charges q' and q'' is equal to the sum of these two charges, and

- the the overall force acting on a two-component body is equal to the sum of the forces acting on each of the components.

It is reasonable to require that the two estimates lead to the same number, i.e., that

$$f(q' + q'') = f(q') + f(q'').$$

In general, we have functions that satisfy the following property for all x and y:

$$f(x + y) = f(x) + f(y). \tag{5}$$

Such functions are known as *additive*.

What can we deduce from additivity. Let us consider the case when we have a dependence of one quantity on the other $y = f(x)$. Let us assume that the function $f(x)$ is differentiable. In this case, we can differentiate both side of equation (5) with respect to y and then substitute $y = 0$. As a result, we first get $\dfrac{df}{dx}(x + y) = \dfrac{df}{dy}(y)$, and then $\dfrac{df}{dx}(x) = c$, where we denoted $c \stackrel{\text{def}}{=} f'(0)$. Integrating both sides of the formula $\dfrac{df}{dx}(x) = c$, we get $f(x) = c \cdot x + c_0$, where c_0 is an integration constant.

For $x = 0$, the formula (5) takes the form $f(0) = 2f(0)$, hence $f(0) = 0$. Thus, $c_0 = 0$, and $f(x) = c \cdot x$.

So, additivity implies the linear dependence $y = c \cdot x$.

Comments.

- This result holds without assuming that the function $f(x)$ is differentiable: it is sufficient to assume that it is continuous (or measurable); see, e.g., [1, 26].

- A similar result holds if we have a dependence on several variables, i.e., if we have a dependence $y = f(x_1, \ldots, x_n)$ which is additive in the sense that for each values $x'_1, x''_1, \ldots, x'_n, x'_n$, if $y' = f(x'_1, \ldots, x'_n)$ and $y'' = f(x''_1, \ldots, x''_n)$, then $y = f(x_1, \ldots, x_n)$, where $x_i = x'_i + x''_i$ and $y = y' + y''$.

Additivity is important in economics as well. Many quantities in economics are additive:

- the overall population of a country is equal to the sum of populations in different provinces,

- the overall GDP of a country is equal to the sum of GDPs of different provinces,

- the overall trade volume of a country is equal to the sum of the trade volume of different provinces, etc.

Thus, if we are interested in estimating the trade volume based on the GDP, we can estimate this trade volume in two ways:

- we can plug in the overall GDP into the corresponding formula,

- or we can use this formula to estimate the trade volume of each province, and then add up the resulting estimates.

It is reasonable to require that these two estimates lead to the same result.

Summary. In this paper, we consider three types of natural symmetries:

- scale-invariance $f(\lambda \cdot x) = C(\lambda) \cdot f(x)$ that leads to the power law:
$$f(x) = A \cdot x^c;$$
- shift-invariance $f(x + x_0) = C(x_0) \cdot f(x)$ that leads to the exponential dependence:
$$f(x) = A \cdot \exp(c \cdot x);$$
- additivity $f(x + y) = f(x) + f(y)$ that leads to the linear dependence:
$$f(x) = c \cdot x.$$

3 How We (Should) Make Decisions: the Notion of Utility

Need to describe human preferences. In the previous section, we talked about numerical economic quantities like population, GDP, income, etc. However, economy is driven by human preferences. So, to adequately describe economic processes, in addition to the above-mentioned numerical characteristics, we must also describe human preferences. How can we do it?

How can we describe human preferences? A natural way to describe human preferences is as follows; see, e.g., [13, 23, 29, 35, 40]. We select two extreme alternatives:

- a very bad alternative A_- which is worse than any of the actual options, and
- a very good alternative A_+ which is better than any of the actual options.

Then, for each value p from the interval $[0, 1]$, we can form a lottery $L(p)$ in which we get A_+ with probability p and A_- with the remaining probability $1 - p$. When $p = 0$, the lottery $L(0)$ is simply equivalent to A_-. The larger p, the better the alternative. Finally, when $p = 1$, we get $A(1) = L_+$.

Thus, we get a continuous scale for describing preferences. For each realistic alternative A, it is better than $L(0) = A_-$ and worse than $L(1) = A_+$: $L(0) < A < L(1)$. Of course, if $L(p) < A$ and $p' < p$, then $L(p') < A$. Similarly, if $A < L(p)$ and $p < p'$, then $A < L(p')$. Thus, one can show that there exists a threshold value u such that:

- for $p < u$, we have $L(p) < A$, and

- for $p > u$, we have $A < L(p)$.

For example, we can take $u = \sup\{p : L(p) < A\}$. This value u is called the *utility* of the given alternative A and is denoted by $u(A)$.

We can reformulate the threshold statement by saying that the alternative A is *equivalent* to the lottery $L(u)$, where the equivalent has to be understood in the above threshold sense, i.e., equivalently, that $L(u-\varepsilon) < A < L(u+\varepsilon)$ for all $\varepsilon > 0$. In this sense, the utility $u(A)$ can be defined as a probability u for which the alternative A is equivalent to the lottery $L(u)$.

What if we select a different pair A_- and A_+? The numerical value $u(A)$ of utility obtained by the above construction depends on the choice of A_- and A_+. If we select another pair A'_- and A'_+, then, for the same alternative, we will get a different utility value $u'(A)$. What is the relation between $u(A)$ and $u'(A)$?

To answer this question, let us consider the case when $A'_- < A_- < A_+ < A'_+$ – other cases can be treated similarly. In this case, since A_- and A_+ are between A'_- and A'_+, we can find a utility $u'(A_-)$ and $u'(A_+)$ of each of them with respect to the pair (A'_-, A'_+). Then:

- A_- is equivalent to a (A'_-, A'_+)-lottery $L'(u'(A_-))$, in which we get A'_+ with probability $u'(A_-)$ and A'_- with the remaining probability $1-u'(A_-)$, and

- A_+ is equivalent to a (A'_-, A'_+)-lottery $L'(u'(A_+))$, in which we get A'_+ with probability $u'(A_+)$ and A'_- with the remaining probability $1-u'(A_+)$.

Each alternative A with utility $u(A)$ is, by definition of utility, equivalent to a lottery $L(u(A))$ in which we get A_+ with probability $u(A)$ and A_- with probability $1 - u(A)$. Each of the alternatives A_- and A_+ is, as we have just mentioned, itself equivalent to a lottery. Thus, the original alternative A is equivalent to a complex lottery, in which:

- first, we select A_+ with probability $u(A)$ and A_- with the probability $1 - u(A)$, and then,

- depending on what we selected on the first step, we select A'_+ with probability $u'(A_+)$ or $u'(A_-)$ and we select A'_- with the remaining probability.

As a result of this complex lottery, we always get either A'_- or A'_+. The probability to get A'_+ can be computed by adding probabilities corresponding to two different ways of getting A'_+: it is $u(A) \cdot u'(A_+) + (1 - u(A)) \cdot u'(A_-)$. But by definition of a (A'_-, A'_+)-based utility, this probability is exactly the utility $u'(A)$. Thus,

$$u'(A) = u(A) \cdot u'(A_+) + (1 - u(A)) \cdot u'(A_-) =$$

$$u'(A_-) + u(A) \cdot (u'(A_+) - u'(A_i)).$$

Thus, the transformation from the old utility $u(A)$ to the new utility $u'(A)$ follows the same formulas as when we change the starting point and the measuring unit:

- $u'(A_-)$ plays the role of shift x_0, and
- the difference $u'(A_+) - u'(A_-)$ plays the role of the scaling λ.

So, to analyze the formulas involving utility, we can also use concepts of scale- and shift-invariance.

4 How Utility Depends on Money

Utility u is not proportional to money m. It is an empirical fact that utility is not proportional to money. Intuitively, this is easy to understand: when a person has nothing, adding \$10 feels great, but when this person already has \$1000, adding \$10 does not change much.

So, how is utility depending on money?

Natural starting point. In general, as have mentioned, utilities are defined modulo an arbitrary linear transformation, so we can shift them and/or scale them.

For money, there is a natural starting point corresponding to 0 amount, i.e., corresponding to the case when we have no savings and no debts. Without losing generality, let us select a utility function for which this 0-money situation corresponds to 0 utility. Once the starting point is thus fixed, the only remaining utility transformation is scaling $u \to k \cdot u$.

So what is the dependence of $u(m)$? As we have mentioned earlier, the numerical value describing the amount of money depends on the choice of the monetary unit. It is therefore reasonable to require that the formula $u(m)$ describing the dependence of utility u on money m does not change if we simply change the monetary unit.

In precise terms, this means that if select a different monetary unit, i.e., if we consider new numerical values $m' = \lambda \cdot m$, then we will get the exact same dependence $u'(m')$ of utility of money, probably after appropriately re-scaling the utility into $u' = C \cdot u$. We already know that this scale-invariance leads to the power law $u = A \cdot m^c$ – and this is exactly what was experimentally observed, with $c \approx 0.5$ – see, e.g. [17, 28].

5 Probabilistic Choice

Formulation of the problem. The traditional utility-based decision theory assumes that, when faced several times with the same several alternatives, the person would make the same selection. In reality, if we repeatedly offer the same choice to a person, this person will, in general, select different alternatives in different iterations. Specifically, alternatives with low utility will practically never be selected, the alternative with the largest utility value will be selected most frequently, but alternatives whose utility is close to the largest will also be selected sometimes.

In such situations, all we can try to predict is the frequency (probability) with which each alternative is selected.

Analysis of the problem. As we have mentioned, the larger the utility of an alternative a, the higher the probability that this alternative will be selected. Thus, we can say that the probability $p(a)$ of selecting the alternative a is proportional to some monotonic function $f(u)$ of its utility: $p(a) = C \cdot f(u(a))$. The coefficient of proportionality C can be determined from the condition that one of the alternatives is always selected, and thus, the sum of the selections probabilities should be equal to 1: $\sum\limits_b p(b) = C \cdot \sum\limits_b f(u(b)) = 1$, hence

$$C = \frac{1}{\sum\limits_b f(u(b))}, \text{ and } p(a) = \frac{f(u(a))}{\sum\limits_b f(u(b))}.$$

In these terms, the question is: which monotonic function $f(u)$ should we choose?

Let us apply natural symmetries. As we have mentioned, utility is defined modulo an arbitrary shift $u \to u' = u + u_0$. It is reasonable to select the monotonic function $f(u)$ in such a way that the resulting probabilities do not change if we apply such a shift, i.e., if we replace each value $u(a)$ by a shifted value $u'(a) = u(a) + u_0$.

The original probability is proportional to $f(u)$, the shifted one is proportional to $f(u + u_0)$. So, we conclude that the shifted function $f(u + u_0)$ must be proportional to the original one $f(u)$, i.e., that we should have $f(u + u_0) = C(u_0) \cdot f(u)$ for some proportionality coefficient $C(u_0)$.

We already know that this functional equation leads to $f(u) = A \cdot \exp(c_0 \cdot u)$ for some c_0, and thus, to $p(a) = \dfrac{\exp(c_0 \cdot u(a))}{\sum\limits_b \exp(c_0 \cdot u(b))}$ [21]. This is exactly the formula for which D. McFadden received his Nobel Prize in 2011; see, e.g., [32, 33, 44].

Comment. As we have mentioned earlier, utility is determined not only modulo shift, it is also determined modulo an arbitrary scaling $u \to u' = k \cdot u$. Clearly, McFadden's formula is not invariant with respect to scalings. What if instead of shift-invariance we require scale-invariance?

In other words, what if we require that the probabilities $p(a)$ do not change if we replace each utility $u(a)$ with a re-scaled one $u'(a) = k \cdot u(a)$? Similarly to the shift-invariance case, this requirement implies that $f(k \cdot u) = C(k) \cdot f(u)$ for some $C(k)$, and we know that this leads to $f(u) = A \cdot u^c$ for some c and thus, to $p(a) = \dfrac{(u(a))^c}{\sum\limits_b (u(b))^c}$ [21]. This explains the empirical formula described in [16].

6 Decision Making under Interval Uncertainty

Formulation of the problem. If we know the exact utility value $u(a)$ corresponding to each possible action a, then it is reasonable to select the action that leads to the largest possible value of utility.

However, in many practical situations, we do not know the exact consequences of each possible action and therefore, we cannot determine the exact utility value of each action. At best, for each possible action a, we know the bounds on the utility, i.e., we know the interval $[\underline{u}(a), \overline{u}(a)]$ that contains the actual (unknown) utility value. In such situations of interval uncertainty, how should we make a decision?

Analysis of the problem. The simplest case of the above problem is when:

- we have two alternatives;
- for the first alternative, we know the interval $[\underline{u}, \overline{u}]$; and
- for the second alternative, we know the exact utility value u.

Let is fix $\underline{u}$ and $\overline{u}$ and consider different possible values u.

When the value u is small (e.g., when $u < \underline{u}$), the first alternative is clearly better. When the value u is large (e.g., when $\overline{u} < u$), the second alternative is clearly better. Thus, similarly to the definition of utility, there exists a threshold value $u_0(\underline{u}, \overline{u})$ such that:

- when $u < u_0$, the first alternative is better, and
- when $u_0 < u$, the second alternative is better.

In this sense, the interval $[\underline{u}, \overline{u}]$ is equivalent to the threshold value u_0.

Thus, in general, to compare two or more intervals:

- we compute, for each of these intervals $[\underline{u}(a), \overline{u}(a)]$, the corresponding equivalent value $u_0(\underline{u}(a), \overline{u}(a))$, and then
- we select the action a for which this equivalent value is the largest.

So, the remaining problem is how to find the equivalent value $u_0(\underline{u}, \overline{u})$.

Let us use symmetries. As we have mentioned, utility is defined modulo shifts and scalings. It is therefore reasonable to require that the relation $u = u_0(\underline{u}, \overline{u})$ does not change under such transformations, i.e., that:

- this relation be shift-invariant: if $u_0(\underline{u}, \overline{u}) = u$, then for each possible shift Δu, we have $u_0(\underline{u} + \Delta u, \overline{u} + \Delta u) = u + \Delta u$; and
- this relation be scale-invariant: if $u_0(\underline{u}, \overline{u}) = u$, then for each possible scaling $k > 0$, we have $u_0(k \cdot \underline{u}, k \cdot \overline{u}) = k \cdot u$.

Let us denote, by α_H, a utility value $u_0(0,1)$ which is equivalent to the simplest possible interval $[0,1]$. Clearly, since all the possible values from this interval are greater than or equal to 0, the equivalent value should also be better than or equivalent to 0, i.e., we should have $\alpha_H \geq 0$. Similarly, we should have $\alpha_H \leq 1$.

For each pair of values $\underline{u} < \overline{u}$, due to scale-invariance with $k = \overline{u} - \underline{u}$, the equation $u_0(0,1) = \alpha_H$ implies that $u_0(0, \overline{u} - \underline{u}) = (\overline{u} - \underline{u}) \cdot \alpha_H$. Then, shift-invariance with $\Delta u = \underline{u}$ implies that $u_0(\underline{u}, \overline{u}) = \underline{u} + (\overline{u} - \underline{u}) \cdot \alpha_H$. The right-hand side of this formula can be rewritten as

$$u_0(\underline{u}, \overline{u}) = \alpha_H \cdot \overline{u} + (1 - \alpha_H) \cdot \underline{u};$$

see, e.g., [23]. This is exactly the formula for decision making under interval uncertainty for which Leo Hurwicz received his Nobel prize [15, 29]. Thus, Hurwicz's formula can be derived from natural symmetries.

Comment. Hurwicz's formula is known as the *optimism-pessimism criterion*, for the following reason:

- if $\alpha_H = 1$, this means that the person only takes into account the best possible scenario when making a decision; in other words, this person is a complete optimist;
- if $\alpha_H = 0$, this means that the person only takes into account the worst possible scenario when making a decision; in other words, this person is a complete pessimist;
- intermediate values α_H between 0 and 1 mean that the person take into account both best-case and worst-case scenarios.

7 Taking Future Effects into Account When Making a Decision

Formulation of the problem. When making economic decisions, people naturally value future gains as less beneficial that current ones. An option is which a person gets \$1 at time t is clearly worth less that a dollar now. This makes sense, since if we get a dollar now, we can invest it – e.g., deposit it in a savings account – and thus, get a larger amount by time t. This phenomenon is known as *discounting.*

How to take this phenomenon into account? In other words, what is the price $D(t)$ that a person should be willing to pay for the option of getting \$1 at moment t?

Analysis of the problem. To estimate $D(t)$, let us use shift-invariance. Specifically, for any pair of values t and t_0, the quality $D(t + t_0)$ can be estimated in two different ways:

- we can directly estimate the desired quantity as $D(t + t_0)$;

- alternatively, we can take into account that \$1 at moment $t + t_0$ (which is t periods after the moment t_0) is equivalent to $D(t)$ dollars at moment t_0; each dollar at moment t_0 is equivalent to $D(t_0)$ dollars now; thus, $D(t)$ dollars at moment t_0 are equivalent to $D(t_0) \cdot D(t)$ dollars now.

It is reasonable to require that these two estimates coincide, i.e., that

$$D(t + t_0) = D(t_0) \cdot D(t).$$

This formula is a particular case of the general shift-invariance, so we conclude that $D(t) = A \cdot \exp(c \cdot t)$ for some A and c. Substituting this expression into the above formula, we conclude that $A = 1$ and thus, $D(t) = \exp(c \cdot t)$. This is exactly the usual formula for discounting; see, e.g., [8, 14, 18, 19, 20, 30, 31, 39, 46]. Thus, the usual formula for discounting can be derived from natural symmetries.

Comment. In [46], we showed that symmetries can also be used to explain the empirically observed deviations from the usual discounting formula; see [8, 14, 18, 19, 20, 30, 31, 39] for details on these deviations.

8 Group Decision Making

Formulation of the problem. What if a group of people needs to make a joint decision?

To properly answer this question, we also need to take into account that the group may be unable to come to an agreement. The resulting situation is known as the status quo situation.

Analysis of the problem. We can always shift each individual utility so that for the status quo solution, the utility of each participant is 0.

Once this status quo point is fixed, the only possible symmetries are scalings $u_i \to u'_i = k_i \cdot u_i$. It is reasonable to require that the decision criterion does not change under this scaling.

A reasonable idea is to have an objective function that combines n utilities $u_1, \ldots, u_n$ into a single utility value $u = f(u_1, \ldots, u_n)$. As we have analyzed earlier, in this case, scale-invariance implies that $f(u_1, \ldots, x_n) = A \cdot u_1^{c_1} \cdot \ldots \cdot u_n^{c_n}$.

It is also reasonable to require that all there is no prior preference to any of the participants. In precise terms, this means that the decision should not change if we simply rename the participants. With respect to the above objective function, this means that all the coefficients c_i must coincide, so that $f(u_1, \ldots, u_n) = A \cdot (u_1 \cdot \ldots \cdot u_n)^c$.

Maximizing this function is equivalent to maximizing the product

$$u_1 \cdot \ldots \cdot u_n;$$

[25]. This is exactly the bargaining solution proposed by nobelist John Nash [34, 29]. Thus, Nash's solution can also be derived from symmetries.

9 Cobb-Douglas Production Function

Formulation of the problem. If we know the country's overall capital K and overall labor input L, how can we estimate the country's production Y? In other words, what function $f(K, L)$ should we use to estimate Y?

Analysis of the problem. The numerical values of all these quantities – capital, labor, and production – depend on what units we use to measure them. It is therefore reasonable to require that the corresponding model $Y \approx f(K, L)$ does not change if we simply change the corresponding units. In other words, it is reasonable to require that the dependence $f(K, L)$ be scale-invariant.

We already know that scale-invariance implies that $Y = A \cdot K^\alpha \cdot L^\beta$, for some α and β. This is exactly the well-known Cobb-Douglas production function; see, e.g., [7, 45, 26]. Thus, the Cobb-Douglas formula can also be derived from natural symmetries.

10 Gravity Model for Trade

Formulation of the problem. How can we estimate the volume of trade t_{ij} between the two countries i and j? Clearly, the larger each country's GDPs g_i and g_j, the more trade we can expect. Similarly, the smaller the distance r_{ij} between the two countries, the more trade we expect. What will be a good estimate for t_{ij} as a function of g_i, g_j, and r_{ij}: $t_{ij} = f(g_i, g_j, r_{ij})$?

Analysis of the problem. As we have mentioned earlier, we can apply this formula to countries as a whole or to different regions of these countries – and then add up the resulting trade volumes. It is reasonable to require that the resulting estimate for the trade volume should not depend on whether we consider the country as a whole or its regions. This means that the dependence on g_i should be additive: $f(g'_i + g''_i, g_j, r_{ij}) = f(g'_i, g_j, r_{ij}) + f(g''_i, g_j, r_{ij})$. As we have shown, this requirement implies that the function f should be linear in g_i: $f(g_i, g_j, r_{ij}) = g_i \cdot F(g_j, r_{ij})$, for some coefficient $F(g_j, r_{ij})$ depending on g_j and r_{ij}.

Similarly, we can consider the country j as a whole or as a combination of its regions. A similar additivity requirement enables us to conclude that the trade volume should be linear in g_j as well, so $f(g_i, g_j, r_{ij}) = g_i \cdot g_j \cdot H(r_{ij})$ for some function $H(r)$.

To find the function $H(r)$, it is reasonable to take into account that the distance can be measured in different units, and the formula for the trade should not change whether we use kilometers or miles. The resulting scale-invariance implies that $H(r) = A \cdot r^c$ for some A and c. Thus, we arrive at the following formula for the trade volume between the two countries: $t_{ij} = A \cdot g_i \cdot g_j \cdot r_{ij}^c$ [27].

This is exactly the well-known gravity model; see, e.g., [2, 3, 5, 38, 43]. Thus, the gravity model can indeed be derived from natural symmetries.

Comment. The usual gravity model only takes into account the GDPs g_i and g_j of the two countries. What if we also take into account their populations p_i

and p_j? In this case, additivity implies that t_{ij} is linear in g_i and p_i, and it is also linear in g_j and p_j. Thus, the overall dependence is bilinear, i.e., we get the following more complex (and hopefully, more accurate) estimate [27]:

$$t_{ij} = \frac{G_{gg} \cdot g_i \cdot g_j + G_{gp} \cdot g_i \cdot p_j + G_{pg} \cdot p_i \cdot g_j + G_{pp} \cdot p_i \cdot p_j}{r_{ij}^c}.$$

11 Linear ARMAX-GARCH Models

Formulation of the problem. How can we predict the future value X_t of an economic quantity X based on its previous values X_{t-1}, X_{t-2}, ..., and on the values d_t, d_{t-1}, ..., of an external quantity d that affects X? In other words, which function $f(X_{t-1}, X_{t-2}, \ldots, d_t, d_{t-1}, \ldots)$ provides the best estimate for X_t?

Analysis of the problem. In many cases, the quantities X in which we are interested are additive – like GDP. Similarly, the quantities d that affect X are usually additive – e.g., the amount of foreign direct investment. In such cases, it is reasonable to require that the prediction should not depend on whether we consider the country as a whole or as a combination of several inputs, i.e., to require that

$$f(X'_{t-1} + X''_{t-1}, X'_{t-2} + X''_{t-2}, \ldots, d'_t + d''_t, d'_{t-1} + d''_{t-1}, \ldots) =$$

$$f(X'_{t-1}, X'_{t-2}, \ldots, d'_t, d'_{t-1}, \ldots) + f(X''_{t-1}, X''_{t-2}, \ldots, d''_t, d''_{t-1}, \ldots).$$

We know that this additivity requirement implies that the function f is linear, i.e., that

$$X_t \approx \sum_{i=1}^{p} \varphi_i \cdot X_{t-i} + \sum_{i=1}^{b} \eta_i \cdot d_{t-i},$$

for appropriate coefficients φ_i and η_i.

To get an even more accurate prediction, it is desirable to take into account how accurately this model predicted the past values of X_t, i.e., what were the differences ε_{t-1}, ε_{t-2}, ..., between the actual values and the predictions. For additive quantities and linear models, the differences are also additive, so we get a more accurate linear model

$$X_t \approx \sum_{i=1}^{p} \varphi_i \cdot X_{t-i} + \sum_{i=1}^{b} \eta_i \cdot d_{t-i} + \sum_{i=1}^{q} \theta_i \cdot \varepsilon_{t-i},$$

for some θ_i.

Taking into account that the inaccuracy of this model is exactly what we denoted by ε_t, we this conclude that

$$X_t = \sum_{i=1}^{p} \varphi_i \cdot X_{t-i} + \sum_{i=1}^{b} \eta_i \cdot d_{t-i} + \varepsilon_t + \sum_{i=1}^{q} \theta_i \cdot \varepsilon_{t-i};$$

see, e.g., [36].

This is exactly the AutoRegressive-Moving-Average model with eXogenous inputs (ARMAX) [6, 9]. Thus, this model can indeed be justified by the corresponding symmetries.

Comment. If we denote the standard deviation of ε_t by σ_t, then similar arguments – based on the fact that for independent random variables, variance σ_t^2 is additive – show that the dynamics of standard deviations σ_t is described by a linear formula

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{\ell} \beta_i \cdot \sigma_{t-i}^2 + \sum_{i=1}^{k} \alpha_i \cdot \varepsilon_{t-i}^2;$$

see, e.g., [36]. This is exactly the Generalized AutoRegressive Conditional Heterosckedasticity (GARCH) model [4, 6, 9]. Thus, GARCH formulas also follow from the natural symmetries.

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