Physics's Need for Interval Uncertainty and How It Explains Why Physical Space Is (at Least) 3-Dimensional

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Abstract

One of the fundamental problems of modern physics is the problem of divergence: e.g., when we try to compute the overall energy of the electric field generated by a charged elementary particle, we get a physically meaningless infinite value. In this paper, we show that one way to avoid these infinities is to take into account that measurements are always imprecise – and thus, we never get the exact values of the physical quantities, only intervals of possible values. We also show that 3-dimensional space is the simplest one in which such interval uncertainty is inevitable. This may explain why the physical space is (at least) 3-dimensional.

1 Divergence Problem in Physics

Divergence problem: a brief reminder. The known divergence problem (see, e.g., [5, 13]) is related to the following simple question: what is the mass of an electron? More generally, what is mass of a charged elementary particle – or, even more generally, the mass of any elementary particle which is involved in interactions of some type?

In general, this mass m consists of two parts:

- the mass m_0 of the particle itself, and

- the mass $m_f = \dfrac{E_f}{c^2}$ which is equivalent to the total energy E_f of the electric (or other) field generated by the particle:

$$m = m_0 + m_f.$$

According to special relativity, an elementary particle, i.e., a particle that cannot be further subdivided, is a point-wise object – otherwise, if it was not

point-wise, different spatial locations within the same particle would be, in general, different, since they cannot instantaneously interact. For a point-wise particle, the electric field at a distance r from the particle is proportional to r^{-2}:

$$E = c_E \cdot r^{-2}.$$

The energy density ρ of the field is proportional to E^2, thus, $\rho = c_\rho \cdot r^{-4}$, for an appropriate constant c_ρ. The overall energy E_f of the electric field can be obtained by integrating $\rho(x)$ over the whole 3-D space:

$$E_f = \int \rho(x) \, d^3 x = \int c_\rho \cdot r^{-4} \, d^3 x = \int_{r=0}^{r=\infty} c_\rho \cdot r^{-4} \cdot 4\pi \cdot r^2 \, dr.$$

Thus,

$$E_f = c_f \cdot \int_0^\infty \frac{dr}{r^2} = c_f \cdot \frac{1}{r} \bigg|_{r=0}^{r=\infty},$$

where $c_f \stackrel{\text{def}}{=} c_\rho \cdot 4\pi$.

At $r = 0$, the expression $\dfrac{1}{r}$ tends to infinity, so we conclude that $E_f = \infty$ and thus, that the overall mass $m = m_0 + \dfrac{E_f}{c^2}$ of the electron is ... infinite.

Divergence is caused only by scale-invariance. The divergence problem does not disappear if we change the formula for the dependence of the field on distance and/or change the dimension – as long as we keep the formulas scale-invariant in the following sense.

The standard formulas for electromagnetic interactions (and for many other interactions) do not have a preferred unit of length. As a result, formulas should not change if we simply change the unit of length – e.g., replace meters with feet. If we replace the original unit with a unit which is λ times smaller, then all the numerical values of distance are multiplied by λ, i.e., each distance r in the old units becomes $r' \stackrel{\text{def}}{=} \lambda \cdot r$ in the new units.

Of course, the dependence $y(r)$ of a relevant physical quantity y (the field itself or the field's energy density) on the distance r cannot be literally the same: otherwise, from the condition that $y(\lambda \cdot r) = y(r)$ we should be able to conclude that $y(r) = \text{const}$. The reason for this impossibility is that measuring units for different quantities are often related: e.g., when we change units for distance d, to preserve the formulas like $d = v \cdot t$, where v denotes velocity and t denotes times, we should accordingly change the unit for velocity v.

In our case, for each λ, we not only replace x with $x' = \lambda \cdot x$, we also replace the numerical y with the appropriately re-scaled value $y' = c(\lambda) \cdot y$. We can now require that when $y = f(x)$, then $y' = f(x')$. Substituting the expressions for x' and y' into this formula, we conclude that $f(\lambda \cdot x) = c(\lambda) \cdot f(x)$. For continuous (even for measurable) functions $f(x)$ this equality implies that $f(x) = A \cdot x^c$ for some real numbers A and c; see, e.g., [1]. In particular, scale-invariance implies the power law dependence of the energy density $\rho(r)$ on the distance r from the particle: $\rho(r) = A \cdot r^c$.

In an m-dimensional space, for a spherically symmetric function $\rho(r)$ (i.e., a function depending only on r), we thus have

$$E_f = \int \rho(r)\, d^m x = A \cdot \int r^c\, d^m x = A \cdot \text{const} \cdot \int_0^\infty r^c \cdot r^{m-1}\, dr =$$

$$\text{const} \cdot \int_0^\infty r^{c+m-1}\, dr = \text{const} \cdot \left. \frac{r^{c+m}}{c+m} \right|_0^\infty .$$

This expression is always infinite:

- when $c + m > 0$, it is infinite for $r = \infty$;

- when $c + m < 0$, the above expression is infinite for $r = 0$;

- finally, when $c + m = 0$, the integral of $r^{c+m-1} = r^{-1}$ is equal to $\ln(r)$, and is, therefore, infinite both at $r = 0$ and at $r\infty$.

Thus, the divergence problem indeed follows from scale-invariance, and it does not depend on what exactly is the dependence on the field on the distance or on what is the dimension of the physical space.

How the divergence problem is solved now. The usual solution to the divergence problem is *renormalization*. Crudely speaking, we take $m_0 = -\infty$, so that the sum of minus infinity m_0 and plus infinity $m_f = \dfrac{E_f}{c^2}$ is finite. To be more precise, instead of considering point-wise particles, we consider particles of a finite radius ε. In this case, the overall energy $E_f(\varepsilon)$ is finite. We then take $m_0(\varepsilon) = m - \dfrac{E_f(\varepsilon)}{c^2}$, where m is the empirically observed electron's mass. In the limit $\varepsilon \to 0$, the proper mass $m_0(\varepsilon)$ tends to $-\infty$, the energy $E_f(\varepsilon)$ tends to $+\infty$, but their sum remains equal to the same constant m.

Problem with renormalization. From the mathematical viewpoint, renormalization solves the divergence problem. However, from the physical viewpoint, renormalization looks like a trick.

It is therefore desirable to come up with a more physically meaningful way to avoid infinities. This is what we do in this paper.

2 Physics's Need for Interval Uncertainty

Main idea: interval uncertainty. In explaining the divergence problem, we implicitly assumed that each physical quantity can be, in principle, measured with any possible accuracy – and thus, that each quantity can be characterized by an exact number.

In practice, measurements also have some imprecision, and it is reasonable to consider the case when we cannot perform measurements beyond a certain level of accuracy. In this case, we will never know the exact value of the corresponding quantity; we will only know the interval containing this quantity.

Let us see how this possibility affects the divergence.

Comment. Our analysis of the effect of interval uncertainty on divergence expands the idea presented in [7].

Interval uncertainty helps avoid divergence. Let us assume the distance r can only be measured with accuracy r_0 (and is, thus, defined only with this accuracy). In this case, when the measured distance is r, the actual distance can take all possible values from $r - r_0$ (to be more precise, from $\max(r - r_0, 0)$ since the distance is always non-negative) to $r + r_0$.

The expression $\rho(x) = c_\rho \cdot r^{-4}$ for the dependence of energy density ρ on the distance r is decreasing. Thus, the density $\rho(x)$ of the energy field can take any value from $\underline{\rho}(x) = c_\rho \cdot (r + r_0)^{-4}$ to $\overline{\rho}(x) = c_\rho \cdot (\max(0, r - r_0))^{-4}$. Therefore, the total energy E_f of the particle's electric field lies between the values

$$\underline{E}_f = \int c_\rho \cdot (r + r_0)^{-4} \, d^3x = 4\pi \cdot c_\rho \cdot \int_0^\infty r^2 \cdot (r + r_0)^{-4} \, dr$$

and

$$\overline{E}_f = \int c_\rho \cdot (\max(0, r - r_0))^{-4} \, d^3x = 4\pi \cdot c_\rho \cdot \int_0^\infty r^2 \cdot (\max(0, r - r_0))^{-4} \, dr.$$

The upper bound $\overline{E}_f$ is infinite, but the lower bound $\underline{E}_f$ is finite, since the lower density is bounded by a constant in the vicinity of $r = 0$. Thus, the fact that we have an infinite upper bound does not imply that the energy is infinite: it can be finite, as long as it is larger than or equal to $\underline{E}_f$.

In short, in the presence of interval uncertainty, there is no divergence.

Comments.

- It is easy to see that the divergence at $r = 0$ disappears for all possible power law dependencies $\rho(x) \sim r^c$ and for all possible spatial dimensions.

- In addition to a fixed-accuracy interval uncertainty, we can also consider a more realistic model, in which we have different accuracies with different degree of confidence. As a result, for each quantity, instead of a single interval, we have a nested family of intervals corresponding to which degree of confidence. This is, in effect, a fuzzy set; see, e.g., [4, 6, 8, 10, 11, 12, 14]. In this fuzzy case, we also eliminate divergence.

3 Why Physical Space Is at Least 3-Dimensional

Role of probabilities. According to quantum physics, all quantities can be determined only with some probabilities; see, e.g., [5, 13]. From this viewpoint, the most important quantity is the probability – i.e., in mathematical terms, a probability measure on the set of all possible events.

Probabilities can be interval-valued. Even for the usual Lebesgue measure (length, area, volume, etc.) some sets S are not measurable, meaning that the

lower probability $\underline{P}(S) = \sup\{P(A) : A \subseteq S\}$ differs from the upper probability $\overline{P}(S) = \inf\{P(A) : S \subseteq A\}$. In such cases, we can say that the measure of the set S is the interval $[\underline{P}(S), \overline{P}(S)]$. So, it is possible (and natural) to have interval-valued probabilities – and thus, interval-type values for all other quantities, such as the expected values, moments, etc.

In 2-D case, it is possible to avoid intervals altogether. Interestingly, while interval-valued probabilities are possible in all dimensions, they are not necessarily appearing in 2-D physical space. The reason for this is that in the 2-D space, it is possible to extend Lebsgue measure to a shift- and rotation-invariant finitely-additive measure defined on all possible 2-D sets [2, 9].

In contrast, in 3-D and in higher dimensions, intervals are inevitable. In 3-D space, it is known that we can decompose a unit ball (= filled sphere) into finitely many pieces, and then shift and rotate these pieces so that these shifted-and-rotated pieces form two balls identical to the original one; see, e.g., [3]. This counterintuitive possibility is known as *Banach-Tarski paradox*.

This result shows that in the 3-D case, it is not possible to extend Lebesgue to shift- and rotation-invariant finitely additive measure: otherwise, decompositions and shift would preserve overall measure, while in the context of the paradox, the final volume is twice larger that the original one. A similar construction is possible for all higher dimensions.

This explains why the physical space is at least 3-dimensional. As we have mentioned earlier, to avoid physically meaningless divergences, it is necessary to have interval uncertainty. In 1-D and 2-D cases, it is possible to avoid interval uncertainty – and thus, get divergence. Starting with dimension 3, however, interval values are inevitable and thus, divergence is not possible. This explains why the physical space is at least 3-dimensional – with the usual 3-D physical space being the simplest space with this property.

Acknowledgments

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