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Joule's 19th Century Energy Conservation Meta-Law and the 20th Century Physics (Quantum Mechanics and General Relativity): 21st Century Analysis

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Abstract Joule's Energy Conservation Law was the first "meta-law": a general principle that all physical equations must satisfy. It has led to many important and useful physical discoveries. However, a recent analysis seems to indicate that this meta-law is inconsistent with other principles – such as the existence of free will. We show that this conclusion about inconsistency is based on a seemingly reasonable – but simplified – analysis of the situation. We also show that a more detailed mathematical and physical analysis of the situation reveals that not only Joule's principle remains true – it is actually strengthened: it is no longer a principle that all physical theories *should* satisfy – it is a principle that all physical theories *do* satisfy.

Keywords Joule · Energy Conservation Law · Free will · General Relativity · Planck's constant

1 Introduction

Joule's Energy Conservation Law: historically the first meta-law. Throughout the centuries, physicists have been trying to come up with equations and laws that describe different physical phenomenon. Before Joule, however, there were no general principle that restricted such equations.

James Joule showed, in [10–12], that different physical phenomena are inter-related, and that there is a general principle covering all the phenom-

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ena. Specifically, he showed that energy can be transformed from one type to another – e.g., from mechanical energy to heat, and that in all these transactions, the overall energy is conserved.

This Energy Conservation Law became the first "meta-law", the first general principle that restricts possible physical theories.

Joule's meta-law in the 20 century: it led to important physical discoveries. This meta-principle turned out to be very helpful for working physics: by restricting possible physical theories, it helped find the correct ones.

A classical example of this help is the discovery of neutrinos; see, e.g., [5, 23]. Specifically, it has been known that while combined neutrons and protons form stable atomic nuclei, stand-alone neutrons are not stable: they decay into protons, emitting electrons in the process. The puzzling problem was that the total energy of the resulting proton and electron is smaller than the energy of the original neutron – which seemingly contradicted to the energy conservation law. To preserve this law, Fermi conjectured that a yet unknown particle – which he called "small neutron" (*neutrino* in Italian) was capturing ("stealing") the missing energy, in what was later called an *urca*-process, after the Russian slang word for a small thief. And yes, neutrinos were later found – so the Energy Conservation Law not only survived, one again it proved to be very helpful for working physics.

Joule's meta-law in the 20 century: it naturally follows from symmetries. In the 20th century, Noether's Theorem showed that energy conservation is indeed a fundamental principle – since it follows from the natural idea that in fundamental physical equations, there is no fixed moment of time, and that thus all physical equations should not change if we simply select a different starting point for measuring time [5,16,23].

What we do in this paper: detailed analysis of Joule's meta-law reveals unexpected subtleties. In this paper, we show that from the viewpoint of 20th century physics – quantum mechanics and relativity theory – the situation with energy conservation is not so simple.

First, in Section 2, following [13], we show that if we try to introduce a natural idea of freedom of will into quantum physics, we naturally get *non*-conservation of energy (which, in its turn, as we show in Section 3 – following [15] – leads to Planck's constant becoming a new physical field).

At first glance, this conclusion may seem to "kill" Joule's meta-law. However, the situation is not so simple. First, as we show in Section 4 – following [14] – it is important to distinguish between the usual *mathematical* formulation and the *physical* meaning of energy conservation. To get an adequate physical meaning we need to also take into account relativistic effects. As we show in Section 5, if we relativize a theory of a field or a system whose energy is not conserved, then all we get is a very strong gravitational field – that compensated for the decreased energy of the original field.

In other words, in the relativistic version of the original theory, energy does not disappear, it simply gets transformed into the gravitational energy

– just like in Joule's experiments, mechanical energy and heat energy got transformed into each other. So not only Joule's principle becomes valid again – this principle is strengthened. It is no longer a principle that all physical theories *should* satisfy – it is a principle that all physical theories *do* satisfy.

2 Free Will and Energy Conservation: A Seeming Contradiction

A brief overview of this section. Modern physical theories are deterministic in the sense that:

- once we know the current state of the world,
- we can, in principle, predict all the future states.

This was true for classical (pre-quantum) theories, this is true for modern quantum physics. On the other hand, we all know that we can make decision that change the state of the world – even if, for most of us, a little bit. This intuitive idea of free will permeates all our life, all our activities – and it seems to contradict the determinism of modern physics. It is therefore desirable to incorporate the idea of free will into physical theories. In this paper, we show that in quantum physics, free will leads to non-conservation of energy. This non-conservation is a microscopic purely quantum effect, but it needs to be taken into account in future free-will quantum theories.

Physics is mostly deterministic. Traditionally, in physics, the state of world changes with time in accordance with appropriate differential equations; see, e.g., [5]. For example:

- in Newton's mechanics, we can use Newton's equations;
- to describe the changes in the electromagnetic field, we can use Maxwell's equations;
- to describe the changes in the state ψ of a quantum system, we can use Schrödinger's equations
$$\mathrm{i}\cdot\hbar\cdot\frac{d\psi}{dt}=H\psi, \tag{1}$$
in which H is an operator describing the total energy of the system.

In all these situations, once we know the state of the world at some moment of time t_0, we can uniquely determine its future state.

It is important to take free will into account when describing the physical world. In physics, the future state of the world is pre-determined. This pre-determination contradicts our intuitive understanding that we humans have free will, that often, we can make decisions, and the outcomes of these decisions are not pre-determined: depending on what we decide, the state of the world will change.

Free will is not just an abstract philosophical viewpoint, it is a practical notion that guides our lives and our behavior. It is therefore desirable to modify physics to avoid this disturbing contradiction between physics and our everyday behavior; see, e.g., [1–4, 6–9, 18, 20–22, 24–26] and references therein.

In classical (pre-quantum) physics, it is relatively easy to come up with equations that allow free will. Let us start with the situation in classical (pre-quantum) physics. Let us start with simple physical systems, such as point particles, whose state $s(t)$ at any given moment of time t can be described by describing the values of finitely many quantities $s_1(t), \ldots, s_n(t)$. For example, in the original Newton's approximate description of celestial bodies as points, to describe the state of each body, it is sufficient to describe the current values x_1, x_2, and x_3 of its three spatial coordinates, three components v_1, v_2, and v_3 of the current velocity, and the body's mass m. In electrodynamics of point particles, we need to add electric change q to the list of these quantities. To describe a system of several interacting points, we need to describe the quantities describing each of these points.

Dynamical equations describe how each of these quantities change:

$$\frac{ds_i}{dt} = f_i(s_1, s_2, \ldots).$$

For example, in Newton's celestial mechanics, such equations describe how the corresponding parameters $x_i^{(j)}$, $v_i^{(j)}$, and $m^{(j)}$ of different bodies $j = 1, 2, \ldots$ change:

$$\frac{dx_i^{(j)}}{dt} = v_i^{(j)}; \quad \frac{dm^{(j)}}{dt} = 0;$$

$$\frac{dv_i^{(j)}}{dt} = G \cdot \sum_{k \neq j} \frac{m^{(k)} \cdot (x_i^{(k)} - x_i^{(j)})}{\sqrt{\left(x_1^{(k)} - x_1^{(j)}\right)^2 + \left(x_2^{(k)} - x_2^{(j)}\right)^2 + \left(x_3^{(k)} - x_3^{(j)}\right)^2}},$$

where G is the gravitation constant.

If we take freedom of will into account, then the change in the state $\frac{ds_i}{dt}$ is no longer uniquely determined by the current state $s(t)$. So, to determine the desired change, we also need to describe the values of some other quantities $w_1, \ldots$, which we can set arbitrarily because of our freedom of will:

$$\frac{ds_i}{dt} = f_i(s_1, s_2, \ldots, w_1, \ldots).$$

There is no differential equations for describing how the quantities w_k change, since we can change them at will.

When the effect of the new quantities is small, we get a small change in the original physical theory.

In quantum physics, the situation is drastically different. In quantum physics, the situation is different. In quantum physics, the state of the world at any given moment of time t is described by a wave function $\psi(t)$, and the change in this state is described by Schrödinger's equation (1). In this equation, the change is determined by the Hamilton operator H that describes the total energy of the system.

So, if we want to allow non-determinism, if we want the ability to change the derivative $\dfrac{d\psi}{dt}$, we have to be able to change the Hamilton operator.

How this leads to non-conservation of energy. In quantum case, as we have concluded, freedom of will means that we can modify the Hamilton operator, the operator that described the total energy of the system. What does it mean that the Hamilton operator changes? It means for the some states, the energy value changes. Thus, in effect, in quantum physics, freedom of will means that, by exercising our will, we can change the total energy of the system. In other words, *in quantum physics, free will seems to lead to non-conservation of energy.*

How big is expected energy non-conservation? As we have mentioned earlier, in classical (pre-quantum) effect freedom of will does not necessarily lead to energy non-conservation. Thus, energy non-conservation caused by the freedom of will is a purely quantum effect, that disappears in the classical limit, when the Planck's constant $\hbar$ tends to 0. So, this purely quantum effect should be proportional to $\hbar$ and thus, it should be reasonably small. This smallness explains why this effect have not been observed: in our usual free-will decisions, we control macro-size objects, objects for which the quantum-size microscopic changes in energy are not easy to measure.

Conclusion. To make sure that physics is in better accordance with our intuition and our everyday experience, it is important to incorporate freedom of will into physical theories. Current physical theories are all based on quantum mechanics; it is therefore necessary to incorporate freedom of will into quantum physics. In this section, we show that this incorporation seems to lead to an unexpected observable effect: non-conservation of energy.

This non-conservation is a purely quantum effect, it is microscopically small for macro-objects, but it needs to be taken into account in future free-will quantum theories.

3 Auxiliary Result: If Energy Is Not Conserved, then Planck's Constant Is No Longer a Constant

A brief overview of the section. For any physical theory, to experimentally check its validity, we need to formulate an alternative theory and check whether the experimental results are consistent with the original theory or with an alternative theory. In particular, to check whether energy is conserved, it is necessary to formulate an alternative theory in which energy is not conserved. Formulating such a theory is not an easy task in quantum physics, where the usual Schroedinger equation implicitly assumes the existence of an energy (Hamiltonian) operator whose value is conserved. In this paper, we show that the only way to get a consistent quantum theory with energy non-conservation is to use Heisenberg representation in which operators representing physical quantities change in time. We prove that in this representation, energy is conserved if and only if Planck's constant remains a constant. Thus, an appropriate quantum analogue of a theory with non-conserved energy is a theory in which Planck's constant can change – i.e., is no longer a constant, but a new field.

3.1 Formulation of the Problem

Every physical law needs to be experimentally tested. Physics is a rapidly changing science, new discoveries are being made all the time, experimental discoveries that are often inconsistent with the existing physics and which lead to a development of new physical theories. Testing the existing physical theories is one of the main ways how physics evolves.

How physical laws can be experimentally tested. To test a physical law, we must:

- formulate an alternative theory in which this law is not valid (while others are valid),
- find a testable experimental situation in which the predictions of this alternative theory differ from the predictions of the original theory, and then
- experimentally check which of the two theories is correct.

Example. This is how the General Relativity theory (alternative at that time) was experimentally tested: by experimentally comparing the predictions of Newton's gravitation theory – the prevalent theory of that time – with the predictions of the alternative theory; see, e.g., [17].

How can we test energy conservation law? A problem. One of the fundamental physical laws is the energy conservation law. At first glance, checking this law is easy: even on the level of Newton's physics, with the usual equations of motion

$$\frac{d^2x_i}{dt^2} = \frac{1}{m} \cdot f_i, \tag{2}$$

relating acceleration $\frac{d^2x_i}{dt^2}$ with the force f_i, there are many non-potential force fields $f_i(x)$ in which energy is not conserved.

The problem appears when we take quantum effects into account, i.e., when we consider the quantum equations. In quantum physics, the main equation – originally formulated by Schroedinger – has the form (see, e.g., [5])

$$\mathrm{i} \cdot \hbar \cdot \frac{\partial \psi}{\partial t} = H\psi, \tag{3}$$

where $\mathrm{i} \stackrel{\text{def}}{=} \sqrt{-1}$, $\psi(x)$ is the wave function describing the quantum state, and H is a so-called *Hamiltonian*, an operator describing the energy of a state. In a non-potential force field, there is no well-defined notion of a total energy and thus, it is not possible to write down the corresponding quantum equation.

Discussion. The need to test the energy conservation law on quantum level is not purely theoretical:

– on a pragmatic level, serious physicists considered the possibility of micro-violations of energy conservation starting from the 1920s [5];
– as the previous section shows, on a more foundational level, the intuitive ideas of free will seem to lead to possible energy non-conservation.

What we do in this section. In this section, we show how to form a quantum theory in which energy is not conserved. Specifically, we show that for quantum theories, energy non-conservation is equivalent to changing Planck's constant. Thus, in quantum physics, checking whether energy is conserved is equivalent to checking whether Planck's constant changes.

3.2 Analysis of the Problem

Schroedinger and Heisenberg representations of quantum physics: reminder. In quantum physics (see, e.g., [5]), states are described by elements of a Hilbert space – e.g., of the space of all square-integrable functions $\psi(x)$ – and physical quantities are described by linear operators in this space.

Historically, quantum physics started with a description by Heisenberg, in which states are fixed but operators change. Very soon, it turned out that in most cases, an alternative representation is more computationally advantageous – a representation in which operators are fixed but states change. This representation was originally proposed by E. Schroedinger and is therefore known as the Schroedinger representation.

Since we cannot use Schroedinger's representation, we will use the Heisenberg one. As we have mentioned, the Schroedinger's equation implicitly assumes the existence of (conserved) energy. Thus, to describe situations in which energy is not conserved, it is reasonable to use the Heisenberg representation.

Heisenberg representation: first approximation. In the Heisenberg representation, physical quantities like coordinates x_i and components $p_i \stackrel{\text{def}}{=} m \cdot \dfrac{dx_i}{dt}$ of the momentum vector are represented by operators. In the first approximation, the usual quantum mechanics is described by the usual Newton's equations

$$\frac{dx_i}{dt} = \frac{1}{m} \cdot p_i, \quad \frac{dp_i}{dt} = f_i, \tag{4}$$

with the only difference that instead of scalars x_i and p_i, we now consider operators. This description was first found by P. Ehrenfest (see, e.g., [5]).

The difference between the scalars and operators is that operators, in general, do not commute, i.e., in general, for two operators a and b, we have $[a,b] \stackrel{\text{def}}{=} ab - ba \neq 0$. Specifically, in the usual quantum physics, operators x_i and x_j corresponding to different coordinates commute with each other, operators p_i and p_j commute with each other, but operators x_i and p_i do not commute:

$$[x_i, x_j] = 0, \quad [p_i, p_j] = 0, \quad [p_i, x_j] = \mathrm{i} \cdot \hbar \cdot \delta_{ij}, \tag{5}$$

where the Kronecker's delta δ_{ij} is equal to 1 when $i = j$ and to 0 otherwise. For the usual energy-preserving quantum mechanics, these commuting relations get conserved as the operators x_i and p_i change in time – in accordance with Ehrenfest equations (4).

In the first approximation, the commutator $[a, b]$ can be described in terms of the Poisson brackets (see, e.g., [5]). Namely, for arbitrary functions $a(x, p)$ and $b(x, p)$ of coordinates $x = (x_1, x_2, x_2)$ and momentum $p = (p_1, p_2, p_3)$, we have

$$[a, b] = \mathrm{i} \cdot \hbar \cdot \{a, b\} + o(\hbar), \tag{6}$$

where

$$\{a, b\} \stackrel{\text{def}}{=} \sum_k \left(\frac{\partial a}{\partial p_k} \cdot \frac{\partial b}{\partial x_k} - \frac{\partial a}{\partial x_k} \cdot \frac{\partial b}{\partial p_k} \right). \tag{7}$$

As an example, let us show what happens for the Heisenberg commutator $[a, b]$ for which $a = p_i$ and $b = x_j$. Since $a = p_i$ depends only on p_i, we have $\dfrac{\partial p_i}{\partial p_k} = \delta_{ik}$ and $\dfrac{\partial p_i}{\partial x_k} = 0$. Similarly, since $b = x_j$ depends only on x_j, we have $\dfrac{\partial x_j}{\partial p_k} = 0$ and $\dfrac{\partial x_j}{\partial x_k} = \delta_{jk}$. Thus,

$$\{p_i, x_j\} = \sum_k \delta_{ik} \cdot \delta_{jk} = \delta_{ij}.$$

When the force comes from a potential field, Planck's constant is conserved. Let us show that in the potential field with potential energy $V(x)$, when $f_i = -\dfrac{\partial V}{\partial x_i}$, Planck's constant is conserved. Indeed, let us assume that the commuting relations (5) hold at a certain moment of time t_0. In particular, this means that $[p_i, x_j] = \mathrm{i} \cdot \hbar \cdot \delta_{ij}$. Let us show that – at least in the first approximation – this relation is conserved, in the sense that $\dfrac{d}{dt}[p_i, x_j] = 0$.

First, we should note that since $[a, b] = ab - ba$, we have

$$\frac{d}{dt}([a, b]) = \frac{d}{dt}(ab - ba) = \frac{da}{dt}b + a\frac{db}{dt} - \frac{db}{dt}a - b\frac{da}{dt} = \left[\frac{da}{dt}, b\right] + \left[a, \frac{db}{dt}\right].$$

Thus, we have

$$\frac{d}{dt}([p_i, x_j]) = \left[\frac{dp_i}{dt}, x_j\right] + \left[p_i, \frac{dx_j}{dt}\right]$$

Due to Ehrenfest equations, we have $\dfrac{dp_i}{dt} = f_i$ and $\dfrac{dx_i}{dt} = \dfrac{1}{m} \cdot p_i$, we have

$$\frac{d}{dt}([p_i, x_j]) = [f_i, x_j] + \frac{1}{m} \cdot [p_i, p_j]. \tag{8}$$

Since f_i depend only on the coordinates, and all coordinate operators commute, we have $[f_i, x_j] = 0$. Since all the components of the momentum commute, we have $[p_i, p_j] = 0$. Thus, we conclude that $\dfrac{d}{dt}([p_i, x_j]) = 0$.

Similarly, we can conclude that the second derivative of the Heisenberg commutator is also equal to 0. Indeed, by differentiating both sides of the equation (8), we conclude that

$$\frac{d^2}{dt^2}([p_i,x_j]) = \left[\frac{df_i}{dt},x_j\right] + \frac{1}{m}\cdot[f_i,p_j] + \frac{1}{m}\cdot[f_i,p_j] + \frac{1}{m}\cdot[p_i,f_j]\,. \tag{9}$$

Here, since f_i depends only on coordinates, we have

$$\frac{df_i}{dt} = \sum_\ell \frac{\partial f_i}{\partial x_\ell}\cdot\frac{dx_\ell}{dt} = \frac{1}{m}\cdot\sum_\ell \frac{\partial f_i}{\partial x_\ell}\cdot p_\ell,$$

so

$$\frac{d^2}{dt^2}([p_i,x_j]) = \frac{1}{m}\cdot\left(\left[\sum_\ell \frac{\partial f_i}{\partial x_\ell}\cdot p_\ell, x_j\right] + 2[f_i,p_j] + [p_i,f_j]\right).$$

Thus, to prove that this second derivative is equal to 0, it is sufficient to prove that the expression in parentheses is equal to 0. In the first approximation, this expression is proportional to the sum S of the corresponding Poisson brackets

$$S = \left\{\sum_\ell \frac{\partial f_i}{\partial x_\ell}\cdot p_\ell, x_j\right\} + 2\{f_i,p_j\} + \{p_i,f_j\};$$

so, in the first approximation, it is sufficient to prove that the sum S is equal to 0. In the first bracket, x_j depends only on x_j, so

$$\left\{\sum_\ell \frac{\partial f_i}{\partial x_\ell}\cdot p_\ell, x_j\right\} = \sum_k\sum_\ell \frac{\partial f_i}{\partial x_\ell}\cdot\delta_{k\ell}\cdot\delta_{jk} = \frac{\partial f_i}{\partial x_j}.$$

For the second term of the sum S, since p_j only depends on the momentum, we get

$$\{f_i,p_j\} = -\sum_k \frac{\partial f_i}{\partial x_k}\cdot\delta_{jk} = -\frac{\partial f_i}{\partial x_j}.$$

Similarly,

$$\{p_i,f_j\} = \frac{\partial f_j}{\partial x_i}.$$

Thus, we have

$$S = \frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j}.$$

For the potential field, we have $f_i = -\dfrac{\partial V}{\partial x_i}$ and therefore,

$$\frac{\partial f_i}{\partial x_j} = -\frac{\partial^2 V}{\partial x_i \partial x_j}.$$

Hence, we have

$$S = -\frac{\partial^2 V}{\partial x_i \partial x_j} + \frac{\partial^2 V}{\partial x_i \partial x_j} = 0.$$

What happens when energy is not conserved: an example. We are interested in situations where energy is not conserved. Let us start our analysis with the simplest such situation of the friction force $f_i = -k \cdot v_i$, i.e., force of the type $f_i = -k_0 \cdot p_i$, where $k_0 \stackrel{\text{def}}{=} \frac{k}{m}$. In this case, from the formula

$$\frac{d}{dt}([p_i, x_j]) = [f_i, x_j] + \frac{1}{m} \cdot [p_i, p_j],$$

by using $[p_i, p_j] = 0$, we get

$$\frac{d}{dt}([p_i, x_j]) = -k_0 \cdot [p_i, x_j].$$

In other words, for $h \stackrel{\text{def}}{=} [p_i, x_i]$, we have a differential equation

$$\frac{dh}{dt} = -k_0 \cdot h.$$

From this equation, we conclude that $\frac{dh}{h} = -k_0 \cdot dt$ hence $\ln(h) = \text{const} - k_0 \cdot t$, and $h(t) = \text{const} \cdot \exp(-k_0 \cdot t)$. We know that $h(t_0) = \mathrm{i} \cdot \hbar$, hence

$$h(t) = h(t_0) \cdot \exp(-k_0 \cdot (t - t_0)).$$

At the initial moment t_0, we have $h(t_0) = \mathrm{i} \cdot \hbar$. So, the above equation means, in effect, that Planck's constant $\hbar \stackrel{\text{def}}{=} \frac{[p_i, x_i]}{\mathrm{i}}$ is no longer a constant – it exponentially decreases with time.

Discussion. Let us show that the same phenomenon – of Planck's constant no longer being a constant – occurs for every theory in which energy is not conserved.

3.3 Main Result

Formulation of the main result. Let us consider the general case, when each component f_i of a force is a function of coordinates x and momentum p. We will show that if in the quantum version of this theory, Planck's constant remains a constant, i.e., we have $[p_i, x_j] = \mathrm{i} \cdot \hbar \cdot \delta_{ij}$ for all moments of time, then the field f_i is a potential field, i.e., has the form $f_i = -\frac{\partial V}{\partial x_i}$ for some function $V(x)$.

This means that if f_i is *not* a potential field, then Planck's constant is no longer a constant.

Proof. If Planck's constant is a constant, this means, in particular, that we have $\frac{d}{dt}([p_i, x_j]) = 0$. Explicitly differentiating the left-hand side, we conclude that $[f_i, x_j] + \frac{1}{m} \cdot [p_i, p_j] = 0$. Since $[p_i, p_j] = 0$, we get $[f_i, x_j] = 0$. In the first approximation, this means that the corresponding Poisson bracket is equal to 0: $\{f_i, x_j\} = 0$. Since x_j depends only on the coordinate, we get

$$\{f_i, x_j\} = \sum_k \frac{\partial f_i}{\partial p_k} \cdot \delta_{kj} = \frac{\partial f_i}{\partial p_j} = 0.$$

The fact that all partial derivatives of f_i relative to p_j are equal to 0 means that f_i does not depend on the momentum. In other words, the force f_i depends only on the coordinates x_j.

Now, since we know that f_i depend only on the coordinates, for the second time derivative $\frac{d^2}{dt^2}([p_i, x_j])$, we can repeat arguments from the previous section and conclude that in the first approximation, this second derivative is proportional to

$$S = \frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j}.$$

So, from the fact that the second derivative is equal to 0, we conclude that $S = 0$, i.e., that

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}$$

for all i and j. It is known that these equalities are necessary and sufficient conditions for the existence of a field V for which $f_i = -\frac{\partial V}{\partial x_i}$. Thus, we have proved that f_i is indeed a potential field.

The conclusion has been proven.

Discussion. Theories in which Planck's constant is no longer a constant but a new physical field $s(x)$ have been proposed; see, e.g., [19].

It should be mentioned that we may need to go beyond the proposed theory: indeed, these theories only consider a *scalar* field $s(x)$ corresponding to

$$[p_i, x_j] = \mathrm{i} \cdot \hbar \cdot s(x) \cdot \delta_{ij},$$

while, in general, the commutator $[p_i, x_j]$ can be an arbitrary *tensor*.

4 Analysis of the Problem: We Should Distinguish Between the Usual Mathematical Formulation and the Physical Meaning of Energy Conservation

A brief overview of this section. In most physical theories, total energy is conserved. For example, when the kinetic energy of a particle decreases, the potential energy increases accordingly. For some physical systems, energy is not conserved. For example, if we consider a particle moving with friction, the energy of the particle itself is not conserved: it is transformed into thermal energy of the surrounding medium. For simple systems, energy is easy to define. For more complex physical systems, such a definition is not easy. To describe energy of generic systems, physicists came up with a general notion of energy based on the Lagrangian formalism – a minimal-action representation of physical theories which is now ubiquitous. For many physical theories, this notion leads to physically meaningful definitions of energy. In this section, we show that there are also examples when the Lagrangian-motivated notion of energy is not physically meaningful at all – e.g., according to this definition, all dynamical systems are energy-conserving.

4.1 Energy Conservation: Physical Meaning and Lagrangian-Based Description

Energy conservation: physical meaning. Some physical systems are *conservative* in the sense that their total energy is preserved. For example, the dynamics of a particle in a potential field $V(x) = V(x_1, x_2, x_3)$ is described, in Newtonian mechanics, by Newton's equations

$$m \cdot \ddot{x}_i = -\frac{\partial V}{\partial x_i}, \tag{10}$$

where $\dot{x}_i$, as usual, denotes time derivative. For this particle, the overall energy

$$E = \frac{1}{2} \cdot m \cdot \sum_{i=1}^{3} (\dot{x}_i)^2 + V(x) \tag{11}$$

is conserved: when the kinetic energy $\frac{1}{2} \cdot m \cdot \sum_{i=1}^{3} (\dot{x}_i)^2$ decreases, the potential energy $V(x)$ increases appropriately, and vice versa.

When energy is not conserved: physical meaning. A classical example of a physical system for which energy is not conserved is a system with friction. Its simplest case is when we do not even have any potential field, i.e., when the dynamical equations have the form

$$m \cdot \ddot{x}_i = -k \cdot \dot{x}_i, \tag{12}$$

for some friction coefficient k. This equation can be further simplified into

$$\ddot{x}_i = -k_0 \cdot \dot{x}_i, \tag{13}$$

for $k_0 \stackrel{\text{def}}{=} \frac{k}{m}$. A system that follows this equation slows down, its velocity (and hence, its kinetic energy) exponentially decreases with time – without being transferred into any other type of energy.

From the physical viewpoint, this non-conservation of energy means that the system described by the equation (13) is not closed: the energy lost in this system is captured by other objects. For friction, it is very clear where this energy goes: it gets transformed into the thermal energy, i.e., into kinetic energy of individual molecules in the surrounding medium.

Need to go beyond simple examples. For simple particles, energy is easy to define and easy to analyze. However, for more complex physical systems, especially when fields are involved, it is not easy to find an appropriate expression for energy.

A general Lagrangian approach to energy conservation. Newton's physics was originally formulated in terms of differential equations. It turns out that most physical theories can be equivalently described in terms of the *minimal action* principle: the actual dynamics of particles and fields is the one that minimizes a special physical quantity called *action* S. For particles, action has the form $S = \int L(x(t), \dot{x}(t))\,dt$, where the function $L(x(t), \dot{x}(t))$ is known as the *Lagrangian*. For example, for the Newtonian particle in a potential field $V(x)$, the Lagrangian has the form

$$L = \frac{1}{2} \cdot m \cdot \sum_{i=1}^{3} (\dot{x}_i)^2 - V(x). \tag{14}$$

For fields $f(x)$, ..., the action S has a similar form

$$S = \int L(f(x), \ldots, f_{,i}(x), \ldots)\,dx,$$

where $f_{,i}$ denotes the corresponding partial derivative $f_{,i} \stackrel{\text{def}}{=} \dfrac{\partial f}{\partial x_i}$.

The Lagrange formulation of physical theories is currently ubiquitous. One of the main reasons for this ubiquity is that, according to modern physics, the correct picture of the physical world comes from quantum mechanics. It is not easy to find a quantum analogue of a physical theory based on its system of differential equations, but when a physical theory is given in Lagrangian terms, its quantization is much more straightforward: in the Feynman's integration-over-trajectories formulation, the amplitude $\psi_{A,B}$ of a transition from a state A to the state B is proportional to the "sum" (integral) of the expression $\exp\left(\mathrm{i} \cdot \dfrac{S}{\hbar}\right)$ over all trajectories leading from A to B, and the probability to observe the transition into different states B is proportional to the squared absolute value of this amplitude $|\psi_{A,B}|^2$; see, e.g., [5,16].

Once we know the Lagrangian, we can use Euler-Lagrange equations to derive the corresponding differential equations. For particles, these equations take the form

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = 0. \tag{15}$$

One can easily check that for the Newtonian Lagrangian (14), we get exactly Newton's equations (10). For fields, the equations take the form

$$\frac{\partial L}{\partial f} - \sum_i \frac{\partial}{\partial x_i}\left(\frac{\partial L}{\partial f_{,i}}\right) = 0. \tag{16}$$

In the Lagrange approach, the energy of a particle is formally defined as

$$E_L \stackrel{\text{def}}{=} \sum_i \dot{x}_i \cdot \frac{\partial L}{\partial \dot{x}_i} - L. \tag{17}$$

One can check that for the Newtonian Lagrangian (14), we get the standard expression (11) for energy. A similar expression defines energy for a field theory [5,16]:

$$E_L = \sum_f \sum_i f_{,i} \cdot \frac{\partial L}{\partial f_{,i}} - L. \tag{18}$$

What we do in this section. The Lagrangian approach has been very successful in describing physical energy of different particle and field systems. What we show, however, is that in some simple cases, the Lagrangian formalism does not adequately convey the physical meaning of energy conservation.

4.2 A Simple Example When the Physical Meaning of Energy Conservation Differs from the Lagrangian-Based Energy

Description of the simple example. Let us consider the simplest possible example of a physical system in which, from the physical viewpoint, energy is not conserved: a 1-D particle with friction, whose dynamics is described by the equation

$$\ddot{x}(t) = -k_0 \cdot \dot{x}(t). \tag{19}$$

What we will do. In this subsection, we will show that this system can be described by a Lagrangian and thus, for this system, energy (as defined in the Lagrangian formalism) is well conserved. This will show that – at least on this example – the Lagrangian formalism does not adequately convey the physical meaning of energy conservation.

In the next section, we show that this inadequacy is not a freaky property of this particular simple system: a generic dynamical system describing a 1-D particle can be described by an appropriate Lagrangian.

Towards finding an appropriate Lagrangian. The classical Newtonian Lagrangian (14) is a sum of two terms: a term depending only on $\dot{x}_i$ and a term depending only on x_i. Let us look for a similar type Lagrangian for describing the equation (19), i.e., let us look for a Lagrangian of the type

$$L = a(\dot{x}) + b(x), \tag{20}$$

for some functions $a(\dot{x})$ and $b(x)$. For this Lagrangian, Euler-Lagrange equations (15) lead to

$$b'(x) - \frac{d}{dt} a'(\dot{x}) = 0, \tag{21}$$

where $b'(x)$ and $a'(\dot{x})$, as usual, indicated derivatives of the corresponding functions. By applying the chain rule to the formula (21), we get

$$b'(x) - a''(\dot{x}) \cdot \ddot{x} = 0. \tag{22}$$

We want to find a Lagrangian that leads to differential equation (19). For this Lagrangian, the formula (22) will be true when we substitute the expression (19) for the acceleration $\ddot{x}$. As a result, we get the following formula

$$b'(x) + k_0 \cdot a''(\dot{x}) \cdot \dot{x} = 0, \tag{23}$$

i.e., equivalently,

$$k_0 \cdot a''(\dot{x}) \cdot \dot{x} = -b'(x) \tag{24}$$

for all possible values x and $\dot{x}$.

The left-hand side of the formula (24) does not depend on $\dot{x}$, and its right-hand side does not depend on x. Since these two sides are equal, this means that this expression cannot depend neither on x nor on $\dot{x}$ and is, therefore, a constant. Let us denote this constant by C. Then, from the condition that the right-hand side is equal to this constant, we conclude that $b'(x) = -C$, hence $b(x) = -C \cdot x + C_0$. The constant term C_0 in the Lagrangian does not affect the corresponding equations (15) and can thus be safely ignored. So, we have $b(x) = -C \cdot x$.

Similarly, from the condition that the left-hand side of the formula (24) is equal to the constant C, we conclude that

$$k_0 \cdot a''(y) \cdot y = C, \tag{25}$$

where, for simplicity, we denoted $y \stackrel{\text{def}}{=} \dot{x}$. From (25), we conclude that

$$a''(y) = \frac{C}{k_0 \cdot y}. \tag{26}$$

Integrating over y, we get

$$a'(y) = \frac{C}{k_0} \cdot \ln(y) + C_0, \tag{27}$$

and, integrating once again, that

$$a(y) = \frac{C}{k_0} \cdot y \cdot \ln(y) + C_0 \cdot y + C_1. \tag{28}$$

Ignoring the constant C_1 and taking into account that $L(x, \dot{x}) = a(\dot{x}) + b(x)$ and that $b(x) = -C \cdot x$, we get the following expression for the desired Lagrangian:

Resulting Lagrangian. The system (19) can be described by the Lagrangian

$$L(x, \dot{x}) = \frac{C}{k_0} \cdot \dot{x} \cdot \ln(\dot{x}) + C_0 \cdot \cdot x - C \cdot x. \tag{29}$$

Comment. One can easily check that for this Lagrangian, Euler-Lagrange equations (15) indeed lead to the equations (19).

Resulting expression for conserved "energy". Here,

$$\frac{\partial L}{\partial \dot{x}} = \frac{C}{k_0} \cdot (\ln(\dot{x}) + 1) + C_0.$$

Thus, applying the usual formula (17) to the Lagrangian (29), we get the expression

$$E_L = \dot{x} \cdot \frac{\partial L}{\partial \dot{x}} - L = \frac{C}{k_0} \cdot \dot{x} + C \cdot x. \tag{30}$$

One can easily check that this "energy" is indeed conserved. Indeed, here

$$\frac{dE_L}{dt} = \frac{d}{dt}\left(\frac{C}{k_0} \cdot \dot{x} + C \cdot x\right) = \frac{C}{k_0} \cdot \ddot{x} + C \cdot \cdot x. \tag{31}$$

Substituting the expression $\ddot{x} = -k_0 \cdot \dot{x}$ into this formula, we indeed get

$$\frac{dE_L}{dt} = 0.$$

4.3 From the Simplest Example to a General Dynamical System

What we do in this subsection. One may think that the weird conclusion – that for a friction particle, energy is well-defined and conserved – is caused by the fact that we have selected a very simple dynamical system (10). Alas, this is not the case. Let us show that a similar Lagrangian reformulation is possible for a generic dynamical system

$$\ddot{x}_i = f_i(x_1, \ldots, x_n, \dot{x}_1, \ldots, \dot{x}_n), \quad i = 1, \ldots, n. \tag{32}$$

A simple multi-D case. Let us start with a multi-D analog of a system with friction, in which the differential equations have the form

$$\ddot{x}_i(t) = -k_0 \cdot \dot{x}_i(t). \tag{33}$$

This system can be described, e.g., by a Lagrangian

$$L = \sum_i \frac{1}{k_0} \cdot \dot{x}_i \cdot \ln(\dot{x}_i) - \sum_i x_i. \tag{34}$$

General case. In the general case, differential equations (15) take the form

$$L_{,x_i} - \frac{d}{dt} L_{,\dot{x}_i} = 0, \tag{35}$$

where $L_{,z}$ denotes partial derivative. By using the chain rule to differentiate the expression $L_{,\dot{x}_i}(x_j, \dot{x}_j)$, we get

$$L_{,x_i} - \sum_j L_{,\dot{x}_i x_j} \cdot \dot{x}_j - \sum_j L_{,\dot{x}_i \dot{x}_j} \cdot \ddot{x}_j = 0. \quad (36)$$

Substituting $\ddot{x}_i = f_i$ into this formula and using notations $y_i = \dot{x}_i$, we get

$$L_{,x_i} - \sum_j L_{,y_i x_j} \cdot y_j - \sum_j L_{,y_i \dot{y}_j} \cdot f_j(x_1, \ldots, x_n, y_1, \ldots, y_n) = 0. \quad (37)$$

Our objective is to define a function $L(x_1, \ldots, x_n, y_1, \ldots, y_n)$ of $2n$ variables for which the second-order partial differential equation (37) holds.

Let us show how we can construct such a function. Let us take, e.g., $L(x_1, \ldots, x_n, 0, \ldots, 0) = 0$ when all the derivatives y_i are equal to 0. Then, we extend it to the case when $y_1 \neq 0$ and $y_2 = \ldots = y_n = 0$. With respect to y_1, (37) becomes a simple second order equation

$$\frac{\partial^2 L}{\partial y_1^2} \cdot f_1 + \frac{\partial^2 L}{\partial y_1 \partial \ldots} \cdot \ldots + \ldots = 0,$$

from which one can explicitly obtain such an extension – e.g., by Euler-style step-by-step integration. Then, we can extend this function along y_2, etc. At the end, we get a function defined for all possible values of x_i and y_j.

5 Taking Gravity into Account Resolves the Puzzle

In line with the distinction emphasized by the previous section, let us consider the physical meaning of energy conservation. So far, out of the two main contributions of 20 century to physics – quantum mechanics and relativity – we only took into account quantum mechanics. Let us now take into account relativity as well.

According to Einstein's General Relativity, the equations for the metric tensor field g_{ij} (that describes gravity, i.e., curved space-time) have the following form (see, e.g., [5,17,23]):

$$G_{ij} = T_{ij},$$

where T_{ij} is the stress-energy tensor,

$$G_{ij} \stackrel{\text{def}}{=} R_{ij} - \frac{1}{2} R g_{ij},$$

and R_{ij} and R are special expressions in terms of the components of the metric tensor and their first and second order derivatives.

In terms of the tensor T_{ij}, energy conservation can be expressed as $T^{ij}_{,j} = 0$, where, as before, $a_{,i}$ means partial derivative with respect to the i-th coordinate, and it is implicitly assumed that we add over repeated indices, i.e., in this case, that we actually mean the formula $\sum_j T^{ij}_{,j} = 0$. (It is worth mentioning that the above simplifying sum-less notation was first introduced by Einstein himself and is thus known as Einstein's notations.)

On the other hand, from the definition of the tensor G_{ij}, it follows that $G^{ij}_{;j} = 0$, where, for each object a (scaler or vector or tensor), $a_{;k}$ denotes *covariant derivative*, i.e., derivative of the type $a_{;k} = a_{,k} + \Gamma a$, where Γ is an expression containing g_{ij} and its first derivatives – and which is equal to 0 in non-curved (Minkowski) space-time. Due to Einstein's equation, the formula $G^{ij}_{;j} = 0$ implies that $T^{ij}_{;j} = 0$, i.e., that $T^{ij}_{j} + \Gamma T = 0$. If in the original theory, energy is not conserved, i.e., we have $T^{ij}_{,j} \neq 0$, this means that we have $\Gamma T \neq 0$, i.e., that $\Gamma \neq 0$.

The value $\Gamma = 0$ corresponds to non-curved space-time, so $\Gamma \neq 0$ means that the space-time is curved – i.e., that there is a gravitational field. The larger non-conservation of energy, the larger Γ and thus, the stronger the corresponding gravitational field. Thus, in the relativistic version of the original non-energy-conserving theory, energy does not disappear, it simply gets transformed into the gravitational energy – just like in Joule's experiments, mechanical energy and heat energy got transformed into each other.

So not only Joule's principle becomes valid again – this principle is strengthened. It is no longer a principle that all physical theories *should* satisfy – it is a principle that all physical theories *do* satisfy.

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