How to Generate "Nice" Cubic Polynomials – with Rational Coefficients, Rational Zeros and Rational Extrema: A Fast Algorithm

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Abstract

Students feel more comfortable with rational numbers than with irrational ones. Thus, when teaching the beginning of calculus, it is desirable to have examples of simple problems for which both zeros and extrema point are rational. Recently, an algorithm was proposed for generating cubic polynomials with this property. However, from the computational viewpoint, the existing algorithm is not the most efficient one: in addition to applying explicit formulas, it also uses trial-and-error exhaustive search. In this paper, we propose a computationally efficient algorithm for generating all such polynomials: namely, an algorithm that uses only explicit formulas.

1 Formulation of the Problem

Need for nice calculus-related examples. After students learn the basics of calculus, they practice in using the calculus tools to graph different functions $y = f(x)$. Specifically,

- they find the roots (zeros), i.e., the values where $f(x) = 0$,
- they find the extreme points, i.e., the values where the derivative is equal to 0,
- they find out whether the function is increasing or decreasing between different extreme points – by checking the signs of the corresponding derivatives,

and they use this information – plus the values of $f(x)$ at several points x – to graph the corresponding function.

For this practice, students need examples for which they can compute both the zeros and the extreme points.

Cubic polynomials: the simplest case when such an analysis makes sense. The simplest possible functions are polynomials. For linear functions, the derivative is constant, so there are no extreme point. For quadratic functions, there is an extreme point, but, after studying quadratic equations, students already know how to graph the corresponding function, when it decreases, when it increases. So, for quadratic polynomials, there is no need to use calculus.

The simplest case when calculus tools are needed is the case of cubic polynomials.

To make the materials simpler for students, it is desirable to limit ourselves to rational roots. Students are much more comfortable with rational numbers than with irrational ones. Thus, to make the corresponding example easier for students, it is desirable to start with examples in which all the coefficients, all the zeros, and all the extreme points of a cubic polynomial are rational.

Good news is that when we know that the roots are rational, it is (relatively) easy to find these roots. Indeed, to find rational roots, we can use the *Rational Root Theorem*, according to which for each rational root $x = p/q$ (where p and q do not have any common divisors) of a polynomial $a_n \cdot x^n + a_{n-1} \cdot x^{n-1} + \ldots + a_0$ with integer coefficients $a_0, \ldots, a_{n-1}, a_n$, the numerator p is a factor of a_0, and the denominator q is a factor of a_n; see, e.g., [3].

Thus, to find all the rational roots of a polynomial, it is sufficient:

- to list all factors p of the coefficient a_0,
- to list all factors q of the coefficient a_n, and then
- to check, for each pair (p, q) of the values from the two lists, whether the ratio p/q is a root.

How can we find polynomials for which both zeros and extreme points are rational?

What is known. An algorithm for generating such polynomials was proposed in [1, 2]. This algorithm, however, is not the most efficient one: for each tuple of the corresponding parameter values, it uses exhaustive trial-and-error search to produce the corresponding nice cubic polynomial.

What we do in this paper. In this paper, we produce an efficient algorithm for producing nice polynomials. Namely, we propose simple computational formulas with the following properties:

- for each tuple of the corresponding parameters, these formulas produce coefficients of a cubic polynomial for which all zeros and extreme points are rational, and
- every cubic polynomial with this property can be generated by applying these formulas to an appropriate tuple of parameters.

Thus, for each tuple of parameters, our algorithm requires the same constant number of elementary computational steps (i.e., elementary arithmetic operations) – in contrast with the existing algorithm, in which the number of elementary steps, in general, grows with the values of the parameters.

2 Analysis of the Problem

Let us first simplify the problem. A general cubic polynomial with rational coefficients has the form

$$a \cdot X^3 + b \cdot X^2 + c \cdot X + d. \tag{1}$$

We consider the case when this is a truly cubic polynomial, i.e., when $a \neq 0$.

Roots and extreme points of a function do not change if we simply divide all its values by the same constant a. Thus, instead of considering the original polynomial (1) with four parameters a, b, c, and d, it is sufficient to consider the following polynomial with only three parameters:

$$X^3 + p \cdot X^2 + q \cdot X + r, \tag{2}$$

where

$$p \stackrel{\text{def}}{=} \frac{b}{a}, \quad q \stackrel{\text{def}}{=} \frac{c}{a}, \quad r \stackrel{\text{def}}{=} \frac{d}{a}. \tag{3}$$

When the coefficients a, b, c, and d of the original polynomial (1) were rational, the coefficients of the new polynomial (2) are rational as well; vice versa, if we have a polynomial (2) with rational coefficients, then, for any rational a, we can have a polynomial (1) with rational coefficients $b = a \cdot p$, $c = a \cdot q$, and $d = a \cdot r$. Thus, to find cubic polynomials with rational coefficients, rational roots, and rational extreme points, it is sufficient to consider polynomials of type (2).

We can simplify the problem even further if we replace the original variable X with the new variable

$$x \stackrel{\text{def}}{=} X + \frac{p}{3} \tag{4}$$

for which

$$X = x - \frac{p}{3}. \tag{5}$$

Substituting this expression for X into the formula (2), we get

$$\left(x - \frac{p}{3}\right)^3 + p \cdot \left(x - \frac{p}{3}\right)^2 + q \cdot \left(x - \frac{p}{3}\right) + r =$$

$$x^3 - 3 \cdot \frac{p}{3} \cdot x^2 + 3 \cdot \left(\frac{p}{3}\right)^2 \cdot x - \left(\frac{p}{3}\right)^3 + p \cdot x^2 -$$

$$2 \cdot p \cdot \frac{p}{3} \cdot x + p \cdot \left(\frac{p}{3}\right)^2 + q \cdot x - q \cdot \frac{p}{3} + r =$$

$$x^3 + \alpha \cdot x + \beta, \tag{6}$$

where

$$\alpha = q - \frac{p^2}{3} \tag{7}$$

and

$$\beta = r - \frac{p \cdot q}{3} + \frac{2p^3}{27}. \tag{8}$$

The roots and extreme points of the new polynomial (6) are obtained from the roots and extremes of the original polynomial (2) by shifting by a rational number $p/3$, so they are all rational for the polynomial (6) if and only if they are rational for the polynomial (2).

Describing in terms of roots. Let r_1, r_2, and r_3 denote rational roots of the polynomial (6). Then, we have

$$x^3 + \alpha \cdot x + \beta = (x - r_1) \cdot (x - r_2) \cdot (x - r_3) =$$

$$x^3 - (r_1 + r_2 + r_3) \cdot x^2 + (r_1 \cdot r_2 + r_2 \cdot r_3 + r_1 \cdot r_3) \cdot x - r_1 \cdot r_2 \cdot r_3. \tag{9}$$

By equating the coefficients at x^2, x, and 1 at both sides, we conclude that

$$r_1 + r_2 + r_3 = 0, \tag{10}$$

$$\alpha = r_1 \cdot r_2 + r_2 \cdot r_3 + r_1 \cdot r_3, \tag{11}$$

and

$$\beta = -r_1 \cdot r_2 \cdot r_3. \tag{13}$$

From (10), we conclude that

$$r_3 = -(r_1 + r_2). \tag{14}$$

Substituting the expression (14) into the formulas (11) and (13), we conclude that

$$\alpha = r_1 \cdot r_2 - r_2 \cdot (r_1 + r_2) - r_1 \cdot (r_1 + r_2) = -(r_1^2 + r_1 \cdot r_2 + r_2^2) \tag{15}$$

and

$$\beta = r_1 \cdot r_2 \cdot (r_1 + r_2). \tag{16}$$

Now the polynomial (6) takes the following form:

$$x^3 - (r_1^2 + r_1 \cdot r_2 + r_2^2) \cdot x + r_1 \cdot r_2 \cdot (r_1 + r_2). \tag{17}$$

Using the fact that the extreme points should also be rational. Let us now use the fact that the extreme points should also be rational. Let x_0 denote an extreme point, i.e., a point at which the derivative of the polynomial (17) is equal to 0. Differentiating the expression (17) and equating the derivative to 0, we get

$$3x_0^2 - (r_1^2 + r_1 \cdot r_2 + r_2^2) = 0. \tag{18}$$

The expression in parentheses can be equivalently described as

$$\frac{3}{4} \cdot (r_1 + r_2)^2 + \frac{1}{4} \cdot (r_1 - r_2)^2 = 3y^2 + z^2, \tag{19}$$

where we denoted

$$y \stackrel{\text{def}}{=} \frac{r_1 + r_2}{2} \text{ and } z \stackrel{\text{def}}{=} \frac{r_1 - r_2}{2}. \tag{20}$$

Substituting this expression (20) into the formula (18), we arrive at the following homogeneous quadratic relation with integer coefficients between the rational numbers x_0, y, and z:

$$3x_0^2 - 3y^2 - z^2 = 0. \tag{21}$$

If we divide both sides of equation (21) by y^2, we get a new equation

$$3X_0^2 - 3 - Z^2 = 0, \tag{22}$$

where we denoted $X_0 \stackrel{\text{def}}{=} \dfrac{x_0}{y}$ and $Z \stackrel{\text{def}}{=} \dfrac{z}{y}$. When x_0, y, and z are rational, then X_0 and Z are also rational numbers. Vice versa, when X_0 and Z form a rational-valued solution of the equation (22), then, for any rational number y, by multiplying both sides of equation (22) by y^2, we can get a solution $x_0 = y \cdot X_0$, y, and $z = y \cdot Z$ of the equation (21). Thus, to find all rational solutions of the equation (21), it is sufficient to find al rational solutions of a simplified equation (22).

The simplest solution and the resulting "nice" polynomials. One of the solution of equation (22) is easy to find: namely, when $X_0 = -1$, the equation (22) takes the form $Z^2 = 0$, i.e., $Z = 0$.

This means that for every y, the values $x_0 = -y$, y and $z = 0$ solve the equation (21). The formulas (20) enable us to reconstruct r_1 and r_2 from y and z as

$$r_1 = y + z \text{ and } r_2 = y - z. \tag{23}$$

In our case, this means $r_1 = r_2 = y$. Thus, due to (15) and (16), we have a polynomial $x^3 + \alpha \cdot x + \beta$ with $\alpha = -3y^2$ and $\beta = 2y^3$.

By applying a shift by a rational number s, i.e., by replacing x with $x = X+s$, we transform a "nice" polynomial $x^3 + \alpha \cdot x + \beta$ into a new "nice" polynomial

$$(X + s)^3 + \alpha \cdot (X + s) + \beta = X^3 + 3s \cdot X^2 + (3s^2 + \alpha) \cdot X + (s^3 + \beta + \alpha \cdot s),$$

i.e., a polynomial (2) with $p = 3s$, $q = 3s^2 + \alpha$, and $r = s^3 + \beta$. Finally, by multiplying this polynomial by a rational number a, we get the following family of "nice" polynomials:

$$b = 3a \cdot s, \quad c = a \cdot (3s^2 + \alpha), \quad d = a \cdot (s^3 + \beta + \alpha \cdot s). \tag{24}$$

In our case, with $\alpha = -3y^2$ and $\beta = 2y^3$, we get

$$b = 3a \cdot s, \quad c = a \cdot (3s^2 - 3y^2), \quad d = a \cdot (s^3 + 2y^3 - 3y^2 \cdot s). \tag{24a}$$

Using the general algorithm for finding all rational solutions to a quadratic equation. To find all rational solutions of the equation (21), we will use a general algorithm for finding all rational solutions of a homogeneous quadratic equation with integer coefficients; see, e.g., [4].

We have already found a solution of the equation (22) corresponding to $X_0 = -1$. For this value X_0, the equation (22) has only one solution $(-1, 0)$, for which $X_0 = -1$ and $Z = 0$. Every other solution (X_0, Z) can be connected to this simple solution $(-1, 0)$ by a straight line. A general equation of a straight line passing through the point $(-1, 0)$ is

$$Z = t \cdot (X_0 + 1). \tag{25}$$

When X_0 and Z are rational, the ratio $t = \dfrac{Z}{X_0 + 1}$ is also rational.

Substituting the expression (25) into the equation (22), we get

$$3X_0^2 - 3 - t^2 \cdot (X_0 + 1)^2 = 0,$$

i.e.,

$$3 \cdot (X_0^2 - 1) - t^2 \cdot (X_0 + 1)^2 = 0. \tag{26}$$

Since we consider the case when $X_0 \neq -1$, we thus have $X_0 + 1 \neq 0$. So, we can divide both sides of the equation (26) by $X_0 + 1$ and thus, get the following equation:

$$3 \cdot (X_0 - 1) - t^2 \cdot (X_0 + 1) = 0.$$

From this equation, we can describe X_0 in terms of t: $(3 - t^2) \cdot X_0 = 3 + t^2$, hence

$$X_0 = \frac{3 + t^2}{3 - t^2}. \tag{27}$$

Substituting this expression for X_0 into the formula (25), we conclude that

$$Z = \frac{6t}{3 - t^2}. \tag{28}$$

Towards a general description of all "nice" polynomials. For every rational y, we can now take $x_0 = y \cdot X_0$, y, and

$$z = y \cdot Z = \frac{6yt}{3 - t^2}. \tag{29}$$

Based on y and z, we can compute r_1 and r_2 by using the formulas (23).

We can now use the values r_1 and r_2 from (23) and the formulas (15) and (16) to compute α and β. Since here, $r_1 + r_2 = 2y$, we get

$$\alpha = r_1 \cdot r_2 - (r_1 + r_2)^2 = (y + z) \cdot (y - z) - (2y)^2 = y^2 - z^2 - 4y^2 = -3y^2 - z^2 \tag{30}$$

and

$$\beta = r_1 \cdot r_2 \cdot (r_1 + r_2) = (y^2 - z^2) \cdot (2y) = 2y \cdot (y^2 - z^2). \tag{31}$$

Substituting these expressions for α and β into the formula (24), we get the formulas for computing the coefficients of the corresponding "nice" cubic polynomial:

$$b = 3a \cdot s; \tag{32}$$

$$c = a \cdot (3s^2 + \alpha) = a \cdot (3s^2 - 3y^2 - z^2); \tag{33}$$

$$d = a \cdot (s^3 + \beta + \alpha \cdot s) = a \cdot (s^3 + 2y \cdot (y^2 - z^2) - (3y^2 + z^2) \cdot s). \tag{34}$$

Thus, we arrive at the following algorithm for computing all possible "nice" cubic polynomials.

3 Resulting Algorithm

Here is an algorithm for computing all "nice" cubic polynomials, i.e., all cubic polynomials with rational coefficients for which all three roots and both extreme points are rational.

In this algorithm, we use four arbitrary rational numbers t, y, s, and a. Based on these numbers, we first compute

$$z = \frac{6yt}{3 - t^2}. \tag{29a}$$

Then, we compute the coefficients b, c, and d of the resulting "nice" polynomial (the value a we already know):

$$b = 3a \cdot s; \tag{32}$$

$$c = a \cdot (3s^2 - 3y^2 - z^2); \tag{33a}$$

$$d = a \cdot (s^3 + 2y \cdot (y^2 - z^2) - (3y^2 + z^2) \cdot s). \tag{34a}$$

These expressions cover almost all "nice" polynomials, with the exception of one family of such polynomials, which is described by the formula

$$b = 3a \cdot s, \quad c = a \cdot (3s^2 - 3y^2), \quad d = a \cdot (s^3 + 2y^3 - 3y^2 \cdot s). \tag{24a}$$

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