For Quantum and Reversible Computing, Intervals Are More Appropriate Than General Sets, And Fuzzy Numbers Than General Fuzzy Sets

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Abstract

Need for faster and faster computing necessitates going down to quantum level – which means involving quantum computing. One of the important features of quantum computing is that it is reversible. Reversibility is also important as a way to decrease processor heating and thus, enable us to place more computing units in the same volume. In this paper, we argue that from this viewpoint, interval uncertainty is more appropriate than the more general set uncertainty – and, similarly, that fuzzy numbers (for which all alpha-cuts are intervals) are more appropriate than more general fuzzy sets. We also explain why intervals (and fuzzy numbers) are indeed ubiquitous in applications.

Keywords: reversible computing, quantum computing, interval uncertainty, set uncertainty, fuzzy numbers, fuzzy sets

1 Need for Quantum and Reversible Computing

Need for quantum computing. Our current computers are very fast in comparison with what was available a few years ago. However, no matter how fast the computers, there are always computational tasks – from bioinformatics, from other discipline – that necessitate even faster computers.

To speed up computers, we need to be able to squeeze in more and more memory cells and processing cells into the same volume. For that, we need to make these cells as small as possible. Already, the existing cells contain a small number of molecules. If we decrease them further, they will contain a few molecules and therefore, we will need to take into account quantum effects; see, e.g., [4, 22]. This is exactly the domain of quantum computing – computations that take quantum effects into account; see, e.g., [16].

Quantum computing: additional advantages. Known good news about quantum computing is that, in addition to a speed up caused by microminiaturization of the processing units, we achieve an additional speed up by using innovative algorithms specifically designed for quantum computing. For example, with quantum computers, we can decrease the time needed to find an element in an unsorted array of size n from n to $\sqrt{n}$ computational steps [5, 6, 16]. We can also reduce the time needed to factor large integers of n digits – task needed to decode currently encoded messages – from exponential to polynomial in n [16, 20, 21].

Need for reversible computing. One challenge in designing quantum computers is that on the quantum level, all equations are time-reversible [4, 16, 22], while in the traditional algorithms, even the simplest "and"-operation $a, b \to a \,\&\, b$ is not reversible: if we know its result $a \,\&\, b = 0 =$"false", we cannot uniquely reconstruct the input (a, b).

Reversibility is also important because, according to statistical physics, any irreversible process means increasing entropy, and this leads to heat emission; see, e.g., [4, 22]. Already overheating is one of the reasons why we cannot pack too many processing units into the same volume. So, to pack more, it is desirable to reduce this heat emission – e.g., by using only reversible computations.

2 When Are Algorithms for Data Processing Under Uncertainty Reversible

Need to take uncertainty into account. When are algorithms reversible? We use computers mostly to process data. When processing data, we need to take into account that data comes from measurements, and measurements are never absolutely accurate, the mea-

surement result $\widetilde{x}$ is, in general, somewhat different from the actual value x of the corresponding quantity. It is therefore necessary to take this uncertainty into account when processing data.

Need for interval uncertainty. In many real life situations, the only information that we have about the measurement error $\Delta x \stackrel{\text{def}}{=} \widetilde{x} - x$ is the upper bound Δ on its absolute value: $|\Delta x| \le \Delta$. Once we have a measurement result $\widetilde{x}$, then the only information that can can conclude about the actual value x of the corresponding quantity is that this value is somewhere in the interval $[\widetilde{x} - \Delta, \widetilde{x} + \Delta]$. Such interval uncertainty indeed appears in many practical applications; see, e.g., [7, 10, 11, 15, 18].

Data processing under interval uncertainty. In a data processing algorithm, we take several inputs $x_1, \ldots, x_n$, and we apply an appropriate algorithm to generate the result y depending on these inputs. Let us denote this dependence by $f(x_1, \ldots, x_n)$. In situations when, for each input i, we only know the interval $X_i = [\widetilde{x}_i - \Delta_i, \widetilde{x}_i + \Delta_i]$ of possible values of x_i, then the only information that we can have about y is that y belongs to the set

$$Y = f(X_1, \ldots, X_n) \stackrel{\text{def}}{=}$$

$$\{f(x_1, \ldots, x_n) : x_1 \in X_1, \ldots, x_n \in X_n\}.$$

When the sets X_i are intervals and the function $f(x_1, \ldots, x_n)$ is continuous,the resulting set Y is also an interval.

In most practical situations, the measurement errors are relatively small, so we can expand the function $f(x_1, \ldots, x_n) = f(\widetilde{x}_1 - \Delta x_1, \ldots, \widetilde{x}_n - \Delta x_n)$ in Taylor series and retain only linear terms. Then, we get

$$f(x_1, \ldots, x_n) = f(\widetilde{x}_1 - \Delta x_1, \ldots, \widetilde{x}_n - \Delta x_n) \approx$$

$$\widetilde{y} - \sum_{i=1}^{n} c_i \cdot \Delta x_i,$$

where $\widetilde{y} \stackrel{\text{def}}{=} f(\widetilde{x}_1, \ldots, \widetilde{x}_n)$ and $c_i \stackrel{\text{def}}{=} \dfrac{\partial f}{\partial x_i}_{|x_i = \widetilde{x}_i}$. In other words, the function $f(x_1, \ldots, x_n)$ becomes a linear function of x_i:

$$f(x_1, \ldots, x_n) = \widetilde{y} - \sum_{i=1}^{n} c_i \cdot (\widetilde{x}_i - x_0) = c_0 + \sum_{i=1}^{n} c_i \cdot x_n,$$

where $c_0 \stackrel{\text{def}}{=} \widetilde{y} - \sum\limits_{i=1}^{n} c_i \cdot \widetilde{x}_i$. In other words, data processing can be, in effect, reduced to two operations: multiplication by a constant c_i and addition.

When is this data processing reversible? Multiplication by a constant is always reversible: indeed, if we know the interval $Y = c \cdot X$, then, as one can easily see, we can reconstruct the original interval X as $X = c^{-1} \cdot Y$, i.e., as the set of all the values $c^{-1} \cdot y$, where $y \in Y = c \cdot X = \{c \cdot x : x \in X\}$.

Similarly, addition is also reversible. Indeed, if we know that $x_1 \in [\underline{x}_1, \overline{x}_1]$ and $x_2 \in [\underline{x}_1, \overline{x}_2]$, then:

- the smallest value of $y = x_1 + x_2$ is attained when both x_1 and x_2 are the smallest and is, thus, equal to $\underline{x}_1 + \underline{x}_2$, and
- the largest value of $y = x_1 + x_2$ is attained when both x_1 and x_2 are the largest and is, thus, equal to $\overline{x}_1 + \overline{x}_2$.

Thus, in this case, the interval $Y = [\underline{y}, \overline{y}]$ has the form

$$Y = [\underline{x}_1 + \underline{x}_2, \overline{x}_1 + \overline{x}_2].$$

If we know the interval $Y = [\underline{y}, \overline{y}]$ and we know, e.g., the interval $X_1 = [\underline{x}_1, \overline{x}_1]$, then, from the formulas $\underline{y} = \underline{x}_1 + \underline{x}_2$ and $\overline{y} = \overline{x}_1 + \overline{x}_2$, we can reconstruct $X_2 = [\underline{x}_2, \overline{x}_2]$ as $\underline{x}_2 = \underline{y} - \underline{x}_1$ and $\overline{x}_2 = \overline{y} - \overline{x}_1$.

From interval uncertainty to a more general set uncertainty. In some cases, in addition to knowing that values of x are within a certain interval $[\underline{x}, \overline{x}]$, we also know that some values from this interval are not possible. In this case, the set X of possible values of x is different from an interval.

This set must be bounded. No matter how crude the measurements are, there is always an upper bound Δ on the measurement error. Thus, after each measurement, based on the measurement result $\widetilde{x}$, we can conclude that the set of possible values of the corresponding quantity is located within the interval $[\widetilde{x} - \Delta, \widetilde{x} + \Delta]$ and is, thus, bounded.

This set must be closed. In general, we can safely assume that the set X is closed. Indeed, suppose that x_0 is a limit point of the set, i.e., that for every $\varepsilon > 0$, there are elements $x \in X$ is any ε-neighborhood $(x_0 - \varepsilon, x_0 + \varepsilon)$ of this value x_0.

This means that no matter how accurately we measure the corresponding value, we will not be able to distinguish between the limit value x_0 and a sufficient close value $x \in X$. In other words, no matter how accurately we measure, we will never be able to conclude that the value x_0 is not possible. It is therefore reasonable to simply assume that x_0 is possible. Thus, we can conclude that the set of possible values of each quantity x contains all its limit points, i.e., is closed – since we cannot experimentally distinguish the original set X from its closure.

Data processing under set uncertainty. If we know the set X_1 of possible values of a quantity x_1, and we know the set X_2 of possible values of a quantity x_2, then the set $Y \stackrel{\text{def}}{=} X_1 + X_2$ of possible values of the sum $y = x_1 + x_2$ is equal to

$$Y = \{x_1 + x_2 : x_1 \in X_1 \text{ and } x_2 \in X_2\}.$$

Known result: addition is only reversible for intervals. It is known (see, e.g., [2, 3]) that addition is only reversible for intervals: *if we add any non-interval bounded closed set S to the class of all intervals, additions stops being reversible.*

The proof of this result is very straightforward: if we take $\underline{S} \stackrel{\text{def}}{=} \inf\{x : x \in S\}$ and $\overline{S} \stackrel{\text{def}}{=} \sup\{x : x \in S\}$, then we have

$$[\underline{S}, \overline{S}] + [\underline{S}, \overline{S}] = [\underline{S}, \overline{S}] + S (= [2\underline{S}, 2\overline{S}]),$$

but $[\underline{S}, \overline{S}] \neq S$.

Case of fuzzy uncertainty. In many real-life situations, in addition to the guaranteed upper bound Δ on the absolute value of the measurement error, experts usually know that most probably (or, to be precise, with some high degree of certainty β) measurement errors can be bounded by a smaller bound $\Delta(\beta) < \Delta$. As a result, in addition to the interval $[\widetilde{x} - \Delta, \widetilde{x} + \Delta]$ that is guaranteed to contain the actual value with 100% confidence, we have several narrower intervals $[\widetilde{x} - \Delta(\beta), \widetilde{x} + \Delta(\beta)]$ that contain the actual value x with the corresponding confidences β. In other words, we have a nested family of intervals corresponding to different values β: the larger the β (i.e., the higher the desired confidence), the wider the interval.

Such a family of nested interval is, in effect, an equivalent way of representing a fuzzy number; see, e.g., [1, 8, 12, 13, 14, 17, 23]. If instead of intervals, we have more general sets $S(\beta)$, then we have a *fuzzy set*. The corresponding sets $S(\beta)$ (in particular, intervals) are known as *α-cuts* of the nested-family fuzzy set, where $\alpha \stackrel{\text{def}}{=} 1 - \beta$.

For such fuzzy sets, we can define operations layer-by-layer: for each β (i.e., equivalently, for each α), we process all the sets (or intervals) corresponding to this value β. Since fuzzy numbers correspond to intervals, and general fuzzy sets to general sets, we arrive at the same conclusion as for sets and intervals [2, 3]: *addition is only reversible for fuzzy numbers; if we add any fuzzy set which is not a fuzzy number to the class of all fuzzy numbers, additions stops being reversible.*

3 Good News: Intervals and Fuzzy Numbers Are Ubiquitous – An Explanation

Intervals are ubiquitous. In the previous section, we showed that intervals (and fuzzy numbers) are preferable since they lead to reversible data processing. Interestingly, intervals (and fuzzy numbers) are indeed ubiquitous, they occur much much more frequently in practice as descriptions of uncertainty than any other sets; see, e.g., [18]. Why is that?

A possible explanation: main idea. To understand why intervals are ubiquitous in non-probabilistic uncertainty, let us recall why in probabilistic uncertainty, the most frequently used distributions – normal (Gaussian) ones – are ubiquitous. The usual explanation is that usually, there are many different independent sources of measurement error. As a result, the measurement error is a sum of a large number of small independent random variables each of which corresponds to one of these sources. It is known that in the limit, when the number of terms in such a sum increases, the distribution of the sum tends to normal; this is known as the Central Limit Theorem; see, e.g., [19]. This limit result means that when the number of components is large, the corresponding definition is close to normal. Thus, from the practical viewpoint, we can safely consider the distribution to be normal.

In non-probabilistic case too, the measurement error is the sum of a large number n of small independent error components:

$$\Delta x = \Delta x^{(1)} + \Delta x^{(2)} + \ldots + \Delta x^{(n)}.$$

So, if for each of the components $\Delta x^{(k)}$, we know the set $X^{(k)}$ of possible values, then the set S of possible values of their sum is equal to the sum of these sets:

$$X = X^{(1)} + \ldots + X^{(n)} =$$

$$\{\Delta x^{(1)} + \Delta x^{(2)} + \ldots + \Delta x^{(n)} :$$

$$\Delta x^{(1)} \in X^{(1)}, \ldots, \Delta x^{(n)} \in X^{(n)}\}.$$

It can be shown that, under reasonable conditions, when the number of components increases, the resulting set X also tends to an interval; see, e.g., [9].

Need for a more detailed explanation. The asymptotic closeness is good, but for practical applications, it is desirable to know exactly how close is the resulting set X to an interval. This is what we will analyze in this section.

What does closeness of sets mean: a brief reminder. For every positive real number $\varepsilon > 0$, two

points a and b are *ε-close* is $|a-b| \le \varepsilon$. It is therefore reasonable to say that the sets A and B are *ε-close* if:

- every point $a \in A$ is ε-close to some point $b \in B$, and
- every point $b \in B$ is ε-close to some point $a \in A$.

The smallest value ε with this property is known as the *Hausdorff distance* $d_H(A,B)$ between the two sets.

How to measure smallness of a set. The size of a set A can be naturally measured by its *diameter* $\mathrm{diam}(A)$, i.e., the largest possible distance $d(a,a')$ between the two points a, a' from this set. For bounded closed subsets A of a real line, the diameter is simply equal to the difference between its largest point $\sup A$ and $\inf A$: $\mathrm{diam}(A) = \sup A - \inf A$.

Our main result. Now, we are ready to formulate our main result.

Proposition 1. *If* $\mathrm{diam}(A_i) \le \varepsilon$ *for all* $i = 1, \ldots, n$, *then for the sum* $A = A_1 + \ldots + A_n$, *its Hausdorff distance from some interval* I *does not exceed* $\varepsilon/2$*:*

$$d_H(A, I) \le \varepsilon/2.$$

Comment. This bound cannot be improved, as shown by the following result:

Proposition 2. *For every* n, *there exist closed bounded sets* $A_1, \ldots, A_n$ *for which* $\mathrm{diam}(A_i) \le \varepsilon$ *for all* i, *and for whose sum* $A = A_1 + \ldots + A_n$, *for every interval* I, *we have* $d_H(A, I) \ge \varepsilon/2$.

Proof of Proposition 1. Let us show that the desired inequality holds from the interval $[\underline{a}, \overline{a}]$, where:

- $\underline{a} \stackrel{\text{def}}{=} \underline{a}_1 + \ldots + \underline{a}_n$, where $\underline{a}_i \stackrel{\text{def}}{=} \inf A_i$, and
- $\overline{a} \stackrel{\text{def}}{=} \overline{a}_1 + \ldots + \overline{a}_n$, where $\overline{a}_i \stackrel{\text{def}}{=} \sup A_i$.

To prove the desired inequality, we need to show that:

- every point $a \in A$ is $(\varepsilon/2)$-close to some point from the interval $I = [\underline{a}, \overline{a}]$, and
- vice versa, that every point b from the interval $I = [\underline{a}, \overline{a}]$ is $(\varepsilon/2)$-close to some point from the sum set A.

Let us first prove that every point $a \in A$ is $(\varepsilon/2)$-close to some point from the interval $I = [\underline{a}, \overline{a}]$. Indeed, by definition of the sum set A, every point a from this set has the form $a = a_1 + \ldots + a_n$, where $a_i \in A_i$ for all i. Every point $a_i \in A_i$ is bounded by this set's inf and sup:

$$\underline{a}_i = \inf A_i \le a_i \le \sup A_i \le \overline{a}_i.$$

By adding up n such inequalities, and taking into account that:

- $\underline{a} = \underline{a}_1 + \ldots + \underline{a}_n$,
- $a = a_1 + \ldots + a_n$, and
- $\overline{a} = \overline{a}_1 + \ldots + \overline{a}_n$,

we conclude that $\underline{a} \le a \le \overline{a}$, i.e., that the value a actually itself belongs to the interval I. So, in this case, we can take $b = a$, and get $|a - b| = 0 \le \varepsilon/2$.

Let us prove that, vice versa, every point b from the interval I is $(\varepsilon/2)$-close to some point $a \in A$. Indeed, since all A_i are closed sets, they contain their limit points $\underline{a}_i = \inf A_i \in A_i$. Thus, $\underline{a} = \underline{a}_1 + \ldots + \underline{a}_n \in A$.

Since $b \in I$, we have $b \ge \underline{a}$, so b is larger than or equal to some point $a \in A$. Let us define

$$a_0 = \sup\{a \in A : a \le b\}.$$

Since all A_i are closed sets, the sum A of these sets is also closed, so a_0, as a limit of elements from A, also belongs to A. Of course, in the limit, from $a \le b$, we conclude that $a_0 \le b$.

If $a_0 = \overline{a}$, then, from the fact that $a_0 \le b \le \overline{a}$, we conclude that $b = a_0 = \overline{a}$ and thus, $|a_0 - b| = 0 \le \varepsilon/2$.

Let us now consider the remaining case when

$$a_0 < \overline{a} = \overline{a}_1 + \ldots + \overline{a}_n.$$

Since the point a_0 is in A, it means that

$$a_0 = a_1 + \ldots + a_n$$

for some values $a_i \in A_i$. For each i, we have $a_i \le \sup A_i = \overline{a}_i$. The inequality $a_0 < \overline{a}$ implies that we cannot have $a_i = \overline{a}_i$ for all i: otherwise, we would have

$$a_0 = a_1 + \ldots + a_n = \overline{a}_1 + \ldots + \overline{a}_n = \overline{a}.$$

Thus, there exists an i for which $a_i < \overline{a}_i$. Let us denote one such index by i_0; then $a_{i_0} < \overline{a}_{i_0}$.

Let us now consider a new point $\overline{a}_0 \in A$ in forming which we replace a_{i_0} with $\overline{a}_{i_0}$:

$$\overline{a}_0 = a_1 + \ldots + a_{i_0 - 1} + \overline{a}_{i_0} + a_{i_0 + 1} + \ldots + a_n.$$

Here, we have $\overline{a}_0 - a_0 = \overline{a}_{i_0} - a_{i_0}$ and thus, by the definition of the diameter, this difference is smaller

than or equal to the diameter $\mathrm{diam}(A_{i_0})$, which is, in turn, smaller than or equal to ε. Thus,

$$|\overline{a}_0 - a_0| \le \varepsilon.$$

Since a_0 is the largest point from the set A which is smaller than or equal to b, and $\overline{a}_0 > a_0$, we thus conclude that $a_0 \not\le b$, i.e., that $b < \overline{a}_0$. So, we have $a_0 \le b < \overline{a}_0$. Here, the sum of the distances $|b - a_0|$ and $|b - \overline{a}_0|$ is equal to $|\overline{a}_0 - a_0|$ and is, thus, smaller than or equal to ε: $|b - a_0| + |b - \overline{a}_0| \le \varepsilon$. Thus, at least one of these two distances must be smaller than or equal to $\varepsilon/2$ (since otherwise, if they were both larger than $\varepsilon/2$, their sum would be larger than ε).

In each of these two cases, we have a point from the set A (either a_0 or $\overline{a}_0$) which is $(\varepsilon/2)$-close to the given point $b \in I$. The proposition is proven.

Proof of Proposition 2. Let us take

$$A_1 = \ldots = A_n = \{0, \varepsilon\}.$$

Then, as one can easily see,

$$A = A_1 + \ldots + A_n = \{0, \varepsilon, 2\varepsilon, \ldots, n \cdot \varepsilon\}.$$

Let us show, by reduction to a contradiction, that we cannot have $d_H(A, I) < \varepsilon/2$ for any interval I.

Indeed, suppose that such an interval exists. Then, by definition of the Hausdorff distance, for the point $0 \in A$, there exists a point $b_1 \in I$ for which

$$|b_1 - 0| = |b_1| \le d_H(A, I).$$

Then, since $b_1 \le |b_1|$, we have $b_1 \le d_H(A, I)$. Since $d_H(A, I) < \varepsilon/2$, we thus have $b_1 < \varepsilon/2$.

Similarly, for the point $\varepsilon \in A$, there exists a point $b_2 \in I$ for which $|\varepsilon - b_2| \le d_H(A, I)$ and thus, $\varepsilon - b_2 \le d_H(A, I)$ and $\varepsilon - d_H(A, I) \le b_2$. Since $d_H(A, I) < \varepsilon/2$, we thus have $b_2 > \varepsilon - \varepsilon/2 = \varepsilon/2$.

Since the interval I contains two points $b_1 < \varepsilon/2$ and $b_2 > \varepsilon/2$, it contains all the points in between, including the point $b = \varepsilon/2$. However, for this point $b \in I$, the closest points from A are the points 0 and ε for both of which the distance to $b = \varepsilon/2$ is equal to $\varepsilon/2$ and is, thus, larger than $d_H(A, I)$ – which contradicts to the definition of Hausdorff distance, according to which every point $b \in I$ must be $d_H(A, I)$-close to some point from the set A.

This contradiction proves that the inequality $d_H(A, I) < \varepsilon/2$ is impossible and thus, indeed, always $d_H(A, I) \ge \varepsilon/2$. The proposition is proven.

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