Which Fourier Components Are Most Informative: General Idea and Case Studies

Julio Urenda[1,2], Nancy Avila[3],
Nelly Gordillo[4], and Vladik Kreinovich[2]
[1]Department of Mathematical Sciences
[2]Department of Computer Science
[3]Biomedical Engineering Program,
Department of Metallurgical, Materials and Biomedical Engineering
University of Texas at El Paso
500 W. University
El Paso, TX 79968, USA
nsavila@utep.edu, jcurenda@utep.edu, vladik@utep.edu
[4]Department of Electrical and Computer Engineering
Universidad Autónoma de Ciudad Juárez
Ave. del Charro No. 450 Norte
Ciudad Juárez, Chihuahua, México 32310
nelly.gordillo@uacj.mx

Abstract

In many practical situations, the information comes not in terms of the original image or signal, but in terms of its Fourier transform. To detect complex features based on this information, it is often necessary to use machine learning. In the Fourier transform, usually, there are many components, and it is not easy to use all of them in machine learning. So, we need to select the most informative components. In this paper, we provide general recommendations on how to select such components. We also show that these recommendations are in good accordance with two examples: the structure of the human color vision, and classification of lung dysfunction in children.

1 Which Fourier Components Are Most Informative: Formulation of the Problem

Fourier components are ubiquitous. In many practical situations, what we perceive or what we get from measuring instruments is not the original signal, but its Fourier transform. This is the case with our vision: we perceive the image in colors, i.e., by separating it into the corresponding Fourier components; see, e.g., [3, 5]. Similarly, when we hear music, we perceive it as sequence of notes, i.e., components corresponding to different frequencies.

This is true not only for our perceptions, this is also true for many measurement situations. For example, when a radio telescope observes a distant radio-source, the resulting signal is actually the Fourier transform of the original image; see, e.g., [4].

Need to select the most informative components. Images and signals are often difficult to process. In situations when we do not have exact formulas for detecting the desired features, it is often very efficient to apply machine learning techniques. However, it is difficult to directly apply these techniques to hundreds and thousands of data points that form each image or each signal. To be able to successfully apply these techniques, it is therefore desirable to select the most informative Fourier components.

2 Main Idea

First observation: magnitudes of Fourier components, in general, decrease with frequency. For a bounded-in-time signal or bounded-in-space image, the well-known Parceval Theorem states that the mean squared value of the image or signal $x(t)$ is equal to the mean square value of its Fourier transform $\widehat{x}(\omega)$: $\int x^2(t)\,dt = \int |\widehat{x}(\omega)|\;d\omega$. For a bounded signal or image, the integral $\int x^2(t)\,dt$ is finite, thus, the integral $\int |\widehat{x}(\omega)|\;d\omega$ is also finite. This implies that, on average, the absolute value $|\widehat{x}(\omega)|$ of the Fourier transform must decrease (and tend to 0) with frequency – otherwise, if this value did not decrease, the integral would be infinite.

Resulting first recommendation. The smaller the Fourier component, the less and less easy to distinguish it from the inevitable noise. Thus, the most informative component is the one which is the largest in (absolute) value – and thus, has the largest signal-to-noise ratio. Since, in general, the Fourier components decrease with frequency, a reasonable idea is to select the Fourier component $\widehat{x}(\omega)$ that corresponds to the smallest possible frequency ω_0.

In general, Fourier components are complex numbers $\widehat{x}(\omega) = r(\omega) + \mathrm{i}\cdot i(\omega)$, where $\mathrm{i} \stackrel{\text{def}}{=} \sqrt{-1}$. In these terms, the recommendation is to select the components $r(\omega_0)$ and $i(\omega_0)$.

What other components should we select: brainstorming. One complex-valued component may be not enough to detect the desired features. What other components should we select?

Let us recall that the real part $r(\omega)$ of the Fourier transform is proportional to $\int x(t) \cdot \cos(\omega \cdot t)\,dt$, and its imaginary part $i(\omega)$ is proportional to

$$\int x(t) \cdot \sin(\omega \cdot t)\,dt.$$

For localized signals located close to some value $t \approx t_0$, we thus get $r(\omega) \approx c \cdot \cos(\omega \cdot t_0)$ and $i(\omega) \approx c \cdot \sin(\omega \cdot t_0)$, for some constant c.

As we have already mentioned, the most informative components are the ones whose absolute value is the largest. Thus, for the real-valued components, the first two most informative components correspond to the values where the cosine is equal to ± 1, i.e., the values $\omega \approx 0$ and $\omega \approx \pi/t_0$. (And an even more informative is the difference between these two components, which is equal to $2c$.) Within this frequency range, from 0 to π/t_0, the most informative value of the imaginary part is when the sine is equal to 1, i.e., the value $\omega \approx 0.5 \cdot \pi/t_0$.

Thus, we arrive at the following general recommendation.

Resulting general recommendation. As the most informative Fourier components, we should take:

- the real and imaginary components $r(\omega_0)$ and $i(\omega_0)$ corresponding to the smallest possible frequency ω_0;
- one more real-valued component $r(\Omega)$ corresponding to some larger frequency Ω (that depends on the signal or image) – or, better yet, the difference $r(\omega_0) - r(\Omega)$, and
- an imaginary component $i\left(\dfrac{\omega_0 + \Omega}{2}\right)$ corresponding to a frequency which is exactly halfway between ω_0 and Ω.

What we will do now. We will show, on two examples, that this reasonable crude approximate recommendation actually leads to good results.

3 First Case Study: Human Color Vision

Discussion. This example is not about machine learning selecting the most informative features – it is about which most informative features biological evolution has selected.

What our recommendation suggests. Specifics of human vision is that in this case, it is difficult to separate real and imaginary parts of the signal. So, in this case, our recommendation means that we should select components corresponding to a smaller frequency ω, to a larger frequency Ω, and to a frequency which is exactly halfway between ω_0 and Ω.

Thus, to get the most informative understanding of images, we should select three equidistant frequencies:

$$\omega_0 < \frac{\omega_0 + \Omega}{2} < \Omega.$$

How human vision system actually works. The human vision system does select three different colors – i.e., three different frequencies [3, 5]:

- red, corresponding to 430–480 THz;
- green, corresponding to 540–580 THz; and
- blue, corresponding to 610–670 THz.

For 430–480 and 610–670, the midpoint is 520–575 THz, which is 4% close to the actual middle range of 540–580 THz.

But is this convincing? One may argue that, of course, the midpoint is somewhere in between the two frequencies, so no wonder it is close to the midpoint between them. By the same logic, one could get the same result if we considered, e.g., wavelength. Let us give it a try. In terms of wavelength:

- red corresponds to 635–700 nm,
- green corresponds to 520-560 nm, and
- blue corresponds to 450–490 nm.

Here, for 635–700 and 450–490, the midpoint is 442.5–595 which is only 6% close to the actual middle range of 520–560 nm. So, indeed, the frequency-based description – motivated by our arguments – is much closer to the actual human vision system.

4 Second Case Study: Classifying Lung Dysfunctions

Formulation of the problem. In this case study, we consider three types of lung dysfunctions: asthma, Small Airway Impairment (SAI), and Possible Small Airway Impairment (PSAI). To correctly classify lung dysfunction in children, a promising idea is to use Impulse Oscillometry System (IOS), where a periodic signal with frequency 5 Hz is added to the airflow coming to the patient, and the resulting outflow is described by its Fourier components $r(f) + \mathrm{i} \cdot i(f)$ corresponding to different frequencies f. Of course, since the signal is periodic, all the frequencies are proportional to 5 Hz [2]. It turns out that the most informative frequencies are between 5 and 20 Hz. Which of the corresponding components should we choose?

What our recommendation says. According to our general recommendation, we should select:

- the components $r(5)$ and $i(5)$ corresponding to the smallest possible frequency of 5 Hz,
- the component $r(f)$ corresponding to the largest of the most informative frequencies – in this case, 20 Hz (or, better yet, the difference between the corresponding components $r(5) - r(20)$), and
- the component $i(f)$ corresponding to the midpoint between 5 and 20 Hz.

In this case, the midpoint between 5 and 20 is 12.5 Hz. There is no component with exactly this frequency, but there are two closest frequencies (which are equally close to 12.5 Hz): 10 Hz and 15 Hz.

Thus, according to our general recommendation, the most informative Fourier components should be $r(5)$, $r(20)$ (or, better yet, $r(5) - r(20)$), $i(5)$, $i(10)$, and $i(15)$.

Empirical data is in perfect accordance with our recommendation. To test our recommendation, we used the data collected by our colleague Erika Meraz; see [1] for details. This data contained data sets from 112 patients with known diagnoses.

For each of components of the Fourier transform, we tested how well this component can differentiate between two different diagnoses. Specifically, we evaluated the importance of each component by comparing the means of the two diagnoses to determind if they were statistically different (at the usual confidence level $p < 0.05$). We then selected the components that statistically significantly differentiated every pair of diagnoses – and these were exactly the components mentioned above: $r(5)$, $r(20)$ (or, better yet, the difference $r(5) - r(20)$), $i(5)$, $i(10)$, and $i(15)$; see [1] for details.

Acknowledgments

This research was partially supported:

- by the National Council of Science and Technology of México (CONACYT) via a doctoral fellowship for Nancy Avila,
- by the Research Program on Migration and Health (PIMSA), a program funded by the University of California, Berkeley, Health Initiative of the Americas (HAI), and
- by the US National Science Foundation via grant HRD-1242122 (Cyber-ShARE Center of Excellence).

The data for this research were acquired by Erika Meraz as part of the NIH grant #1 S11 ES013339-01A1: "UTEP-UNM HSC ARCH Program on Border Asthma" (Pilot 1) granted to Dr. Homer Nazeran.

References

[1] N. Avila, H. Nazeran, E. Meraz, N, Gordillo, and C. Aguilar, "Characterization of impulse oscillometric measures of respiratory small airway function in children", *Advances in Electrical and Electronic Engineering*, to appear.

[2] S. Bickel, J., Popler, B. Lesnick, and N. Eid, "Impulse oscillometry: interpretation and practical applications", *Chest*, 2014, Vol. 146, No. 3, pp. 841–847.

[3] R. W. G. Hunt, *The Reproduction of Colour*, Wiley, Chichester, UK, 2007.

[4] G. L. Verschuur and K. I. Kellermann, *Galactic and Extra-Galactic Radio Astronomy*, Springer Verlag, Berlin, Heidelberg, New York, 2012.

[5] G. Wyszecki and W. S. Stiles, *Colour Science: Concepts and Methods, Quantitative Data and Formulae*, Wiley, New York, 2000.