Nonlinear Mechanical Properties of Road Pavements: Geometric Symmetries Explain the Empirical Difference between Roads Built on Clay vs. Granular Soils

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Abstract

It is empirically known that roads built on clay soils have different nonlinear mechanical properties than roads built on granular soils (such as gravel or sand). In this paper, we show that this difficult-to-explain empirical fact can be naturally explained if we analyze the corresponding geometric symmetries.

1 Formulation of the Problem

Mechanical properties of road pavements are non-linear. In the traditional elasticity theory (see, e.g., [6]), it is usually assumed that the elastic materials satisfy Hooke's law, i.e., that the strain (= relative change in the distances) is a linear function of stress (= force per unit area), and vice versa. The corresponding coefficient of stress/strain is called the *modulus* E.

In reality, the Hooke's law is approximate. In most cases – e.g., for buildings – this approximation works very well, since the forces are reasonably small and thus, we can safely ignore terms which are quadratic or of higher order with respect to the corresponding stresses. The only cases when we need to take possible nonlinearity into account are the extreme situations, such as a strong earthquake or a hurricane.

In contrast, on the roads, stresses are high and dynamic: e.g., each passing heavy trucks leads to a huge stress concentrated in a small area. As a result, when describing mechanical properties of road pavements, we need to take non-linear terms into account.

How this nonlinearity is described. Several formulas have been proposed to describe these properties. At present, the empirical comparison between different models seems to indicate that one of these models is the most adequate. To describe this so-far most adequate formula, let us introduce the corresponding notations. The stress corresponding to the force in the i-th direction as applied to a surface which is orthogonal to the j-th direction is denoted by σ_{ij}. In mechanics, it is known that the resulting matrix σ_{ij} is symmetric. The eigenvalues of this symmetric 3×3 matrix are called *principal stresses* and denoted by σ_1, σ_2, and σ_3.

The numerical values of the components σ_{ij} of the stress matrix depend on the selection of the coordinate system. However, based on these components, we can form combinations of these values which are invariant, i.e., do not depend on the coordinate system. These invariants are:

- the trace $\mathrm{Tr}(\sigma) = \sum\limits_{i=1}^{3} \sigma_{ii} = \sigma_1 + \sigma_2 + \sigma_3$ of the matrix; it is called the *bulk stress* and denoted by θ, and
- the trace of the square of this matrix $\mathrm{Tr}(\sigma^2)$, where $(\sigma^2)_{ij} = \sum\limits_{k=1}^{3} \sigma_{ik} \cdot \sigma_{kj}$.

The second invariant is thus equal to

$$\mathrm{Tr}(\sigma^2) = \sum_{i=1}^{3} \sum_{j=1}^{3} \sigma_{ij} \cdot \sigma_{ji} = \sum_{i=1}^{3} \sigma_i^2.$$

When the stress is isotropic, i.e., $\sigma_1 = \sigma_2 = \sigma_3 = \frac{1}{3} \cdot \theta$, then

$$\mathrm{Tr}(\sigma^2) = 3 \cdot \sigma_i^2 = 3 \cdot \left(\frac{1}{3} \cdot \theta\right)^2 = \frac{1}{3} \cdot \theta^2.$$

So, usually, instead of the second invariant, engineers use the difference d between the second invariant and the corresponding isotropic value

$$d = \mathrm{Tr}(\sigma^2) - \frac{1}{3} \cdot \theta^2 = \sum_{i=1}^{3} \sigma_i^2 - \frac{1}{3} \cdot \left(\sum_{i=1}^{3} \sigma_i\right)^2.$$

One can check that this difference can be reformulated in a different form

$$d = \frac{1}{3} \cdot ((\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2),$$

a form in which it is very clear that in the isotropic case, when all the principal stresses are equal to each other, this difference is equal to 0. The difference d is proportional to the square of the stresses. To make it easier to compare numerical values of different invariant characteristics, in mechanics, usually, instead of d, a different characteristic is used which is proportional to the square root of d (and which is, therefore, described in the same units as the principal stresses):

$$\tau_{\rm oct} \stackrel{\rm def}{=} \frac{1}{\sqrt{3}} \cdot \sqrt{d} = \frac{1}{3} \cdot \sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2}.$$

This characteristic is known as the *octahedral shear stress.*

A known inequality describes the relation between the two invariants: namely, we have

$$\tau_{\rm oct}^2 = \frac{1}{9} \cdot (2\sigma_1^2 + 2\sigma_2^2 + 2\sigma_3^2 - 2\sigma_1 \cdot \sigma_2 - 2\sigma_1 \cdot \sigma_3 - 2\sigma_2 \cdot \sigma_3)$$

while

$$\theta^2 = (\sigma_1 + \sigma_2 + \sigma_3)^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 + 2\sigma_1 \cdot \sigma_2 + 2\sigma_1 \cdot \sigma_3 + 2\sigma_2 \cdot \sigma_3.$$

Thus, we always have $\tau_{\rm oct}^2 \le \frac{2}{9} \cdot \theta^2$. So, we always have $\tau_{\rm oct} \le \frac{\sqrt{2}}{3} \cdot \theta$. Here, $\frac{\sqrt{2}}{3} \approx 0.47 < 0.5$, thus the octahedral shear stress is less than a half of the bulk stress.

The equality is attained only when exactly one of the three principal stresses σ_i is different from 0, and the two others are zeros. On average, the ratio is even smaller that half. To get a ballpark estimate of the average ratio, let us assume that each principal stress σ_i is uniformly distributed on some interval $[0, \sigma]$, and that different principal stresses are independent. Then, the expected value $E[\sigma_i^2]$ is equal to $\frac{1}{3} \cdot \sigma^2$ and for $i \ne j$, we have

$$E[\sigma_i] \cdot \sigma_j] = E[\sigma_i] \cdot E[\sigma_j] = \left(\frac{1}{2} \cdot \sigma\right)^2 = \frac{1}{4} \cdot \sigma^2,$$

hence

$$E[\theta^2] = 3E[\sigma_i^2] + 6E[\sigma_i \cdot \sigma_j] = 3 \cdot \frac{1}{3} \cdot \sigma^2 + 6 \cdot \frac{1}{4} \cdot \sigma^2 = \frac{5}{2} \cdot \sigma^2$$

and similarly

$$E[\tau_{\rm oct}^2] = \frac{1}{9} \cdot \left(6E[\sigma_i^2] - 6 \cdot E[\sigma_i \cdot \sigma_j]\right) = \frac{1}{18} \cdot \sigma^2.$$

Thus, $E[\tau_{\rm oct}^2] = \frac{1}{45} \cdot E[\theta^2]$. So, the mean squared value of the octahedral shear stress $\tau_{\rm oct}$ is equal to $\frac{1}{\sqrt{45}} \approx 0.15$ of the mean squared value of the bulk stress.

The modulus E depends on both invariants θ and $\tau_{\rm oct}$. The most empirically accurate formula has the form

$$E = E(\theta, \tau_{\rm oct}) = k_1' \cdot \left(\frac{\theta}{P_a} + 1\right)^{k_2'} \cdot \left(\frac{\tau_{\rm oct}}{P_a} + 1\right)^{k_3'}. \quad (1)$$

This formula was proposed in Ooi et al. [5], and in Mazari et al. [4], it was shown to be the most adequate model for describing the elastic modulus.

Comment. In [1, 3], we provide a theoretical explanation for this formula.

Clay vs. granular soils. The parameters k'_2 and k'_3 depend only on the material – e.g., whether it is clay or some type of granular soil. In contrast, the parameter k'_1 varies strongly even for the same material – e.g., for gravel, the value of k'_1 depends on how big the grains are, what is their density, etc.

Specifically (see, e.g., Section 5.4 "Mechanical Properties" of the official document [7]):

- for fine-grained material like clay, k'_2 is close to 0, and the most significant nonlinear term corresponds to $k'_3 < 0$; while
- for the granular material, i.e., for gravel, sand, or silt (coarse-grained) soils with little or no clay content, the most significant nonlinear term corresponds to $k'_2 > 0$.

Formulation of the problem and what we do in this paper. The problem is that there seems to be no convincing explanation of the above-described empirical relation between the type of the soil and the relative values of the coefficients k'_2 and k'_3 describing the nonlinearity of the corresponding mechanical properties. In this paper, we show that such an explanation can be obtained if we analyze the related geometric symmetries.

2 Symmetry-Based Geometric Analysis of the Problem

General idea. We are interested in analyzing when perturbations affect the mechanical properties of a system. The mechanical properties of the system are largely determined by the system's geometry, in particular, on its symmetries; see, e.g., [2].

In general, if the perturbations have the same symmetry as the original system, then the system retains its symmetry. For example, if we isotropically squeeze a spherical rubber ball from all sides, it remains spherical – only its size decreases. In such cases, the mechanical properties remain largely the same.

On the other hand, if perturbations do not have the same symmetry as the system, this may change the system's symmetry and thus, change the system's mechanical properties. For example, if we squeeze the spherical rubber call in one direction, its shape changes from a sphere to an ellipsoid-like shape.

Let us apply this general idea to our case. In our case, the mechanical properties of a system are described by the modulus E.

Clay is isotopic. Thus, if we apply isotropic pressure – i.e., squeeze it isotropically from all sides – its symmetry remains the same, and thus, its mechanical properties should remain the same. In terms of the principal stresses, the case of an isotropic perturbation corresponds to $\sigma_1 = \sigma_2 = \sigma_3$, i.e., to the case when the octahedral shear stress $\tau_{\rm oct}$ is equal to 0. So, if $\tau_{\rm oct} = 0$, then, no matter what is the value of the overall stress θ, the mechanical properties should remain the same, i.e., the modulus E should not change. For the modulus described by the formula (1), this means that the modulus $E(\theta, 0)$ should not depend on θ. Substituting $\tau_{\rm oct} = 0$ into the formula (1), we conclude that

$$E(\theta, 0) = k'_1 \cdot \left(\frac{\theta}{P_a} + 1\right)^{k'_2}.$$

Thus, the fact that the modulus $E(\theta, 0)$ does not depend on 0 means that we must have $k'_2 = 0$ – which is what we observe.

This is, of course, in the ideal case when the medium is perfectly isotropic. In reality, real-life systems are only approximately isotropic, as a result of which the coefficient k'_2 is only approximately equal to 0 – which is exactly what we observe.

In contrast, gravel is not isotropic, it consists of parts of irregular shape – at best, of the elongated shape which is invariant with respect to rotations around the corresponding axis. In this case, practically any perturbation will change the system's symmetry: even when we have a perturbation which has the same symmetry as one of the granules, all other granules have a different symmetry. Thus, we expect that the mechanical properties will change no matter what perturbation we apply.

So, we expect that the value E described by the formula (1) will change when we apply some stresses; in other words, we expect $k'_2 \neq 0$ and $k'_3 \neq 0$. As we have mentioned earlier, the octahedral shear stress is, on average, almost an order of magnitude smaller than the bulk stress. Thus, it is reasonable to expect that for the granular material, the most significant nonlinear term is the term corresponding to the bulk stress, i.e., the term corresponding to k'_2 – which is also exactly what we observe.

So, symmetries indeed explain the empirical difference between the mechanical properties of fine-grained materials like clay and coarse-grained materials.

3 Remaining Open Problem

In this paper, we provided a *qualitative* explanation for the the empirical difference between the nonlinear mechanical properties of the roads built on clay vs. granular soils. This may be a theoretically interesting result, but, honestly, it does not help practitioners: all we do is explain a formula which they already use. From the practical viewpoint, it is desirable to extend our qualitative analysis to a *quantitative* one, hopefully enabling us to estimate the values k'_2 and k'_3 based on the geometry of the soil particles.

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