In Alsina et al. Derivation of Min-Max Fuzzy Logic from Distributivity, All Conditions Are Necessary: A Proof

Vladik Kreinovich[1], Ildar Z. Batyrshin[2], Olga Kosheleva[1], and Nailya Kubysheva[3]
[1]University of Texas at El Paso, 500 W. University
El Paso, TX 79968, USA, olgak@utep.edu, vladik@utep.edu
[2]CIC, Instituto Politécnico Nacional, CDMX, 07738, Mexico
batyr1@gmail.com
[3]Kazan Federal University, Kazan, 420008, Russia

Abstract

In their 1983 paper, C. Alsina, E. Trillas, and L. Valverde proved that distributivity, monotonicity, and boundary conditions imply that the "and"-operation is min and the "or"-operation is max. In this paper, we show that all these conditions are necessary for Alsina et al. result to be true.

1 Alsina et al. Result: Reminder

In [1], it has been proven that for two binary operations $\vee : [0,1] \times [0,1] \to [0,1]$ and $\& : [0,1] \times [0,1] \to [0,1]$, distributivity, monotonicity, and boundary conditions imply that $a \vee b = \max(a,b)$ and $a \,\&\, b = \min(a,b)$; see also [2, 3, 4, 5].

Let us formulate this result in precise form. In our formulation, we deviate slightly from [1]; namely:

- we will consider the derivations of $\vee$ and $\&$ separately;
- we divide boundary conditions into conditions on $\vee$ and $\&$; and
- we use slightly weaker boundary conditions: e.g., $a \le a \vee 0$ instead of the original $a = a \vee 0$.

Derivation of max. We consider the following conditions:

- (P1) for all a, b, and c, we have $a \,\&\, (b \vee c) = (a \,\&\, b) \vee (a \,\&\, c)$ (*distributivity*);
- (P2) for all a, a', b, and b', if $a \le a'$ and $b \le b'$, then $a \vee b \le a' \vee b'$ (*monotonicity*);

- (P3) for all a, we have $a \,\&\, 1 = 1 \,\&\, 1 = a$; (*first boundary condition*);
- (P4) for all a, we have $a \le a \vee 0$ and $a \le 0 \vee a$ (*second boundary condition*).

Comment. Actually, it is sufficient to consider distributivity only for $b = c = 1$.

Proposition 1. *For every pair of binary operations, if the conditions (P1)-(P4) are satisfied, then $a \vee b = \max(a, b)$.*

Proof. Due to (P4), we have $1 \le 1 \vee 0$. Since $1 \vee 0 \in [0, 1]$, we conclude that $1 \vee 0 = 1$. Due to monotonicity, $1 = 1 \vee 0 \le 1 \vee 1$. Thus, $1 \vee 1 = 1$.

For $b = c = 1$, the distributivity condition implies that for all a, we have $a \,\&\, (1 \vee 1) = (a \,\&\, 1) \vee (a \,\&\, 1)$. Since $1 \vee 1 = 1$, this means

$$(a \,\&\, 1) = (a \,\&\, 1) \vee (a \,\&\, 1).$$

Due to the first boundary condition, this implies that $a \vee a = a$.

If $a \le b$, monotonicity implies that $a \vee b \le b \vee b = b$. On the other hand, due to monotonicity and to the property (P4), we have $b \le 0 \vee b \le a \vee b$. So, $b \le a \vee b \le b$, thus $a \vee b = b$.

Similarly, if $b \le a$, then monotonicity implies that $a \vee b \le a \vee a = a$. On the other hand, due to monotonicity and to the property (P4), we have $a \le a \vee 0 \le a \vee b$. So, $a \le a \vee b \le a$, thus $a \vee b = a$.

In both cases, we have $a \vee b = \max(a, b)$. The proposition is proven.

Derivation of min**.** A similar result proves that the "and"-operation is equal to min. For this purpose, we consider the following conditions:

- (Q1) for all a, b, and c, we have $a \vee (b \,\&\, c) = (a \vee b) \,\&\, (a \vee c)$ (*distributivity*);
- (Q2) for all a, a', b, and b', if $a \le a'$ and $b \le b'$, then $a \,\&\, b \le a' \,\&\, b'$ (*monotonicity*);
- (Q3) for all a, we have $a \vee 0 = 0 \vee a = a$; (*first boundary condition*);
- (Q4) for all a, we have $a \,\&\, 1 \le a$ and $1 \,\&\, a \le a$ (*second boundary condition*).

Comment. Actually, it is sufficient to consider distributivity only for $b = c = 0$.

Proposition 2. *For every pair of binary operations, if the conditions (Q1)-(Q4) are satisfied, then $a \,\&\, b = \min(a, b)$.*

Proof. Due to (Q4), we have $0 \,\&\, 1 \le 0$. Since $0 \,\&\, 1 \in [0, 1]$, we conclude that $0 \,\&\, 1 = 0$. Due to monotonicity, $0 \,\&\, 0 \le 0 \,\&\, 1 = 0$. Thus, $0 \,\&\, 0 = 0$.

For $b = c = 0$, the distributivity condition implies that for all a, we have $a \vee (0 \,\&\, 0) = (a \vee 0) \,\&\, (a \vee 0)$. Since $0 \,\&\, 0 = 0$, this means $(a \vee 0) = (a \vee 0) \,\&\, (a \vee 0)$. Due to the first boundary condition, this implies that $a \,\&\, a = a$.

If $a \le b$, monotonicity implies that $a = a \,\&\, a \le a \,\&\, b$. On the other hand, due to monotonicity and to the property (Q4), we have $a \,\&\, b \le a \,\&\, 1 \le a$. So, $a \le a \,\&\, b \le a$, thus $a \,\&\, b = a$.

Similarly, if $b \le a$, then monotonicity implies that $b = b \,\&\, b \le a \,\&\, b$. On the other hand, due to monotonicity and to the property (Q4), we have $a \,\&\, b \le 1 \,\&\, b \le b$. So, $b \le a \,\&\, b \le b$, thus $a \,\&\, b = b$.

In both cases, we have $a \,\&\, b = \min(a, b)$. The proposition is proven.

Comment. As one can see from the proofs, the propositions are valid not only for the binary operations on the interval $[0, 1]$, but also for binary operations on any linearly ordered set with the smallest element 0 and the largest element 1.

2 Let Us Prove in Both Results, All Four Conditions Are Needed

Derivation of "or"-operations. Let us start with Proposition 1 that derives the max operation. For each of the conditions (P1)–(P4), we will have an example of two operations that satisfy the remaining three conditions and for which the operation $a \vee b$ is different from $\max(a, b)$.

What is we do not require the property (P1). In this case, we can simply take $a \vee b = a + b - a \cdot b$ and $a \,\&\, b = a \cdot b$. One can easily check that in this case, we have monotonicity and both boundary conditions.

What is we do not require the property (P2). Let us consider the following two operations:

- if $a < 1$ and $b < 1$, then $a \,\&\, b = 0$, otherwise $a \,\&\, b = \min(a, b)$;
- if $a = b$ then $a \vee b = a$ else $a \vee b = a + b - a \cdot b$.

One can easily see that for these two operations, the boundary conditions (P3) and (P4) are satisfied: $a \,\&\, 1 = 1 \,\&\, a = a$ and $a \le a \vee 0 = 0 \vee a$. Let us show that these two operations satisfy the distributivity property (P1). To prove this, we will consider all possible cases.

First, we consider the case when $a = 1$. In this case, $a \,\&\, b = 1 \,\&\, b = b$, $a \,\&\, c = 1 \,\&\, c = c$, and $a \,\&\, (b \vee c) = 1 \,\&\, (b \vee c) = b \vee c$, so the distributivity property turns into a trivial equality $b \vee c = b \vee c$.

To complete the proof, it is thus sufficient to consider only the cases when $a < 1$. In such cases:

- it is possible that both values b and c are smaller than 1,
- it is possible that one of these two values is smaller than 1, and
- it is possible that both b and c are equal to 1.

We will consider these three options one by one.

- In the situation when $a < 1$, $b < 1$, and $c < 1$, we have $b \vee c < 1$, thus $a \,\&\, (b \vee c) = a \,\&\, b = a \,\&\, c = 0$. So distributivity turns into the equality $0 = 0 \vee 0$ – which is true for our selection of the "or"-operation.

- In the situation when $a < 1$ and one of the two values b, c is equal to 1 and another is smaller than 1, we can, without losing generality, assume that $b = 1$ and $c < 1$. In this case, $1 = b \le b \vee c \le 1$ implies that $b \vee c = 1$. Thus, $a \,\&\, (b \vee c) = a \,\&\, 1 = a$, $a \,\&\, b = a$, and $a \,\&\, c = 0$. So, the distributivity properties takes the form $a = a \vee 0$, which is indeed true for the selected "or"-operation.

- Finally, in the situation when $a < 1$ and $b = c = 1$, due to $1 = b \le b \vee c \le 1$, we have $b \vee c = 1$. Thus, we have $a \,\&\, (b \vee c) = a \,\&\, 1 = a$, $a \,\&\, b = a \,\&\, c = a \,\&\, 1 = a$, and the distributivity property turns into $a = a \vee a$, which is also true for the selected "or"-operation.

In all the cases, distributivity is proven.

What is we do not require the property (P3). Let us take $a \,\&\, b = 0$ for all a and b, and $a \vee b = a + b - a \cdot b$. In this case, distributivity takes the form $0 = 0 \vee 0$, which is, of course, always true, and we clearly have monotonicity (P2) and the second boundary condition (P4).

What is we do not require the property (P4). Let us take $a \,\&\, b = a \vee b = \min(a, b)$. In this case, we have distributivity, we have monotonicity (P2), and we have the first boundary condition (P3) – i.e., $a \,\&\, 1 = 1 \,\&\, a = a$.

Derivation of "and"-operations. Let us now consider Proposition 2 that derives the min operation. For each of the conditions (Q1)–(Q4), we will have an example of two operations that satisfy the remaining three conditions and for which the operation $a \,\&\, b$ is different from $\min(a, b)$.

What is we do not require the property (Q1). In this case, we can simply take $a \vee b = a + b - a \cdot b$ and $a \,\&\, b = a \cdot b$. One can easily check that in this case, we have monotonicity and both boundary conditions.

What is we do not require the property (Q2). Let us consider the following two operations:

- if $a > 0$ and $b > 0$, then $a \vee b = 1$, otherwise $a \vee b = \max(a, b)$;

- if $a = b$ then $a \,\&\, b = a$ else $a \,\&\, b = a \cdot b$.

One can easily see that for these two operations, the first and the second boundary conditions are satisfied: $a \vee 0 = 0 \vee a = \max(a, 0) = a$ and $a \,\&\, 1 = 1 \,\&\, a \le a$. Let us show that these two operations satisfy the distributivity property. To prove this, we will consider all possible cases.

First, we consider the case when $a = 0$. In this case, $a \vee b = 0 \vee b = b$, $a \vee c = 0 \vee c = c$, and $a \vee (b \,\&\, c) = 0 \vee (b \,\&\, c) = b \,\&\, c$, so the distributivity property turns into a trivial equality $b \,\&\, c = b \,\&\, c$.

To complete the proof, it is thus sufficient to consider only the cases when $a > 0$. In such cases:

- it is possible that both values b and c are positive,
- it is possible that one of these two values is positive, and
- it is possible that both b and c are equal to 0.

We will consider these three options one by one.

- In the situation when $a > 0$, $b > 0$, and $c > 0$, we have $b \,\&\, c > 0$, thus $a \,\&\, (b \vee c) = a \,\&\, b = a \,\&\, c = 1$. So distributivity turns into the equality $1 = 1 \,\&\, 1$ – which is true for our selection of the "and"-operation.
- In the situation when $a > 0$ and one of the two values b, c is equal to 0 and another is positive, we can, without losing generality, assume that $b = 0$ and $c > 0$. In this case, $0 \le b \,\&\, c \le b = 0$ implies that $b \,\&\, c = 0$. Thus, $a \vee (b \,\&\, c) = a \vee 0 = a$, $a \vee b = a \vee 0 = a$, and $a \vee c = 1$. So, the distributivity properties takes the form $a = a \,\&\, 1$, which is indeed true for the selected "and"-operation.
- Finally, in the situation when $a > 0$ and $b = c = 0$, due to $0 \le b \,\&\, c \le b = 0$, we have $b \,\&\, c = 0$. Thus, we have $a \vee (b \,\&\, c) = a \vee 0 = a$, $a \vee b = a \vee c = a \vee 0 = a$, and the distributivity property turns into $a = a \,\&\, a$, which is also true for the selected "or"-operation.

In all the cases, distributivity is proven.

What is we do not require the property (Q3). Without the first property, we can have $a \vee b = 1$ for all a and b, and $a \,\&\, b = a \cdot b$. In this case, distributivity takes the form $1 = 1 \,\&\, 1$ which is, of course, always true, and we clearly have monotonicity (Q2) and the second boundary condition (Q4).

What is we do not require the property (Q4). Let us take $a \,\&\, b = a \vee b = \max(a, b)$. In this case, we have distributivity, we have monotonicity (Q2), and we have the first boundary condition (Q3) – i.e., $a \vee 0 = 0 \vee a = a$.

Conclusion. In both cases, we have shown that each of the four conditions is necessary for deriving min and max.

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