Geometric Explanation for an Empirical Formula Describing Our Galaxy's Warping

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Abstract

In the first approximation, the shape of our Galaxy – as well as the shape of many other celestial bodies – can be naturally explained by geometric symmetries and the corresponding invariances. As a result, we get the familiar shape of a planar spiral. A recent more detailed analysis of our Galaxy's shape has shown that the Galaxy somewhat deviates from this ideal shape: namely, it is not perfectly planar, it is somewhat warped in the third dimension. In this paper, we show that the empirical formula for this warping can also be explained by geometric symmetries and invariance.

1 Formulation of the Problem

In the first approximation, geometry explains shapes of celestial bodies such as galaxies. In modern physics, symmetries – including geometric symmetries – and related invariances play a very important role; see, e.g. [2, 7].

As part of this general trend, it is known that the geometric shapes of most celestial bodies can be explained by symmetry groups and corresponding invariances; see, e.g., [3, 4, 5]. Namely, in the beginning, the Universe was homogeneous, isotropic, and scale-invariant – i.e., in geometric terms, it was invariant with respect to shifts, rotations, and homotheties. Such highly symmetric matter distributions are, however, unstable, so spontaneous symmetry breaking leads us to states with fewer invariances. In general, the more invariances are preserved, the more probable the corresponding transition. Thus, the most probable transition from the original fully symmetric state is to a planar shape ("pancake", a typical shape of a proto-galaxy), and the next most probably transition is to a planar logarithmic spiral – a generic planar shape with a 1-dimensional symmetry group. In perfect accordance with this symmetry analysis, the shape of our Galaxy is indeed well approximated by a planar logarithmic spiral.

Comment. It should be mentioned that eventually, each celestial body gets transformed into the most stable state – of a sphere or, in case of rotation, of a rotating ellipsoid [3, 4, 5].

A recent empirical formula for a more detailed description of our Galaxy's shape. A recent more detailed analysis of the Galaxy's shape [6] has shown that this shape somewhat deviates from the above symmetric form. Specifically, the spiral shape is still there, but the Galaxy is not exactly planar. In the corresponding cylindrical coordinates (r, φ, z) in which the plane has the form $z = 0$, the z-coordinate of the actual Galaxy is only equal to 0 up to a certain threshold distance r_d from the center ($r \le r_d$). For the distances $r > r_d$, the following empirical formula describes the actual shape:

$$z = z_0 \cdot (r - r_d)^2 \cdot \sin(\varphi - \varphi_0), \tag{1}$$

for some parameters z_0 and φ_0.

What we do in this paper. In this paper, we show that this empirical formula can also be naturally explained by the corresponding geometric symmetries and invariances.

2 Analysis of the Problem and the Resulting Explanation

Main idea. Instead of the plane $z = 0$, we have a warped shape. To be more precise, we have a planar shape until $r - r_d$, and then the shape $z(r, \varphi)$ becomes warped.

The original planar shape $z = 0$ is invariant with respect to all rotations

$$\varphi \to \varphi + \alpha$$

around the Galaxy's center. The observed shape (1) is, however, not rotation-invariant. Thus, the corresponding function $z(r, \varphi)$ is not rotation invariant. Since we cannot have a single rotation-invariant function, a natural idea is to assume that the actual shape-describing function $z(r, \varphi)$ belongs to a few-parametric rotation-invariant *family* of functions

$$C_1 \cdot z_1(r, \varphi) + \ldots + C_k \cdot z_k(r, \varphi),$$

where $z_1(r, \varphi), \ldots, z_k(r, \varphi)$ are fixed functions, and $C_1, \ldots, C_k$ are arbitrary coefficients.

Let us select a family with as fewer parameters as possible – since in general, the fewer parameters we need to describe a physical phenomenon, the more convincing our explanation [2]: with a large number of parameters, we can explain anything by properly adjusting the values of these parameters.

Simplest case $k = 1$ does not help. The simplest case if $k = 1$, when we have a family $\{C_1 \cdot z_1(r, \varphi)\}$. For this family, invariance means that if we rotate by an angle α – i.e., if we replace φ with $\varphi + \alpha$ – then we still get a function from the same family, i.e., $z_1(r, \varphi + \alpha) = C(\alpha) \cdot z_1(r, \varphi)$. For each r, we have a functional equation whose solutions are known (see, e.g., [1]): $z_1(r, \varphi) = z_1(r, 0) \cdot \exp(a(r) \cdot \varphi)$ for some $a(r)$. Since rotation by $\alpha = 2\pi$ does not change anything, we should have $z_1(r, \varphi + 2\pi) = z_1(r, \varphi)$. This implies that $a(r) = 0$ and thus, that the corresponding function $z_1(r, \varphi)$ does not depend on the angle φ at all – and we know that it depends.

Thus, to adequately describe the actual shape of our Galaxy, we need to consider families with more parameters – at least $k = 2$.

Case of $k = 2$. In this case, we have a family $\{C_1 \cdot z_1(r, \varphi) + C_2 \cdot z_2(r, \varphi)\}$. The fact that this family in rotation-invariant means that the rotated functions $z_1(r, \varphi + \alpha)$ and $z_2(r, \varphi + \alpha)$ should also belong to this family, i.e., that we should have $z_1(r, \varphi+\alpha) = C_{11}(\alpha) \cdot z_1(r, \varphi) + C_{12}(\alpha) \cdot z_2(r, \varphi)$ and $z_2(r, \varphi+\alpha) = C_{21}(\alpha) \cdot z_1(r, \varphi) + C_{22}(\alpha) \cdot z_2(r, \varphi)$.

From the physical viewpoint, it makes sense to assume that the shapes are smooth and that, that the function $z(r, \varphi)$ is differentiable. Thus, it makes sense to restrict ourselves to the case when both functions $z_i(r, \varphi)$ are differentiable. By using Cramer's formulas, we can explicitly express all four values $C_{ij}(\alpha)$ as rational expressions in terms of the values of the functions $z_i(r, \varphi)$; thus, the corresponding functions $C_{ij}(\alpha)$ are also differentiable. Differentiating the above equations with respect to α and taking $\alpha = 0$, we get, for each r, the following system of linear differential equations with constant coefficients:

$$z_i'(r, \varphi) = c_{i1} \cdot z_1(r, \varphi) + c_{i2} \cdot z_2(r, \varphi) \ (i = 1, 2),$$

where z' denotes derivative with respect to φ and $c_{ij} \stackrel{\text{def}}{=} C_{ij}'(0)$.

It is known that a general solution to such a system is a linear combination of functions $\varphi^m \cdot \exp(a \cdot \varphi) \cdot \sin(b \cdot \varphi + \varphi_0)$. The requirement that $z_i(r, \varphi + 2\pi) = z_i(r, \varphi)$ eliminates powers and exponents and leaves only sines and cosines with integer b. In line with the general argument about minimizing the number of parameters, let us restrict ourselves to the simplest case $b = 1$. Then, for each i, we have $z_i(r, \varphi) = A_i(r) \cdot \sin(\varphi) + B_i(r) \cdot \cos(\varphi)$ for some $A_i(r)$ and $B_i(r)$. The actual function $z_1(r, \varphi)$ is a linear combination of these functions, so we have $z(r, \varphi) = A(r) \cdot \sin(\varphi) + B(r) \cdot \cos(\varphi)$ for appropriate $A(r)$ and $B(r)$.

Which functions $A(r)$ and $B(r)$ should we choose? We know that $z(r, \varphi) = 0$ for $r \le r_d$ and, in general, $z(r, \varphi) \ne 0$ for $r > r_d$. We have assumed that the function is differentiable.

The difference $z(r, \varphi)$ from planarity is relatively small, so for $r > r_d$, it makes sense to expand the dependence $z(r, \varphi)$ in Taylor series and keep only the first few terms in this expansion. We cannot keep only linear terms – otherwise, the resulting piece-wise linear dependence $z(r, \varphi)$ on r will not be differentiable at $r = r_d$. Thus, we need to also keep quadratic terms. One can easily check that the only quadratic functions that smoothly transition to 0 at $r = r_d$ are functions $c \cdot (r - r_d)^2$ for some constant c. Both functions $A(r)$ and $B(r)$ should have this form, for appropriate coefficients c_a and c_b – since they represent $z_1(r, \varphi)$ when $\varphi = \pi/2$ and when $\varphi = 0$. Thus, for $r > r_d$, the above expression for $z(r, \varphi)$ has the form

$$z(r, \varphi) = c_a \cdot (r - r_d)^2 \cdot \sin(\varphi) + c_b \cdot (r - r_d)^2 \cdot \cos(\varphi) =$$
$$(r - r_d)^2 \cdot (c_a \cdot \sin(\varphi) + c_b \cdot \cos(\varphi)). \quad (2)$$

So, we get the desired geometric explanation. The expression $c_a \cdot \sin(\varphi) + c_b \cdot \cos(\varphi)$ can be easily transformed into the form $z_1 \cdot \sin(\varphi - \varphi_0)$, with $z_0 = \sqrt{c_a^2 + c_b^2}$ and appropriate φ_0.

Thus, from the formula (2), we indeed get the desired expression (1).

Remaining open problems. In general, for celestial bodies, geometric invariances do not just explain possible shapes; they also explain relative frequencies of different shapes, prevalent directions of rotation and of the body's magnetic field, etc. [3, 4, 5].

It would therefore be nice to similarly extend the above geometric explanation of our Galaxy's shape to an explanation of other related empirical formulas – starting with other formulas presented in [6].

Acknowledgments

This work was supported in part by the National Science Foundation grants 1623190 (A Model of Change for Preparing a New Generation for Professional Practice in Computer Science) and HRD-1242122 (Cyber-ShARE Center of Excellence).

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