If Space-Time Is Discrete, We May Be Able to Solve NP-Hard Problems in Polynomial Time

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Abstract

Traditional physics assumes that space and time are continuous. However, this reasonable model leads to some serious problems. One the approaches that physicists follow to solve these problems is to assume that the space-time is actually discrete. In this paper, we analyze possible computational consequences of this discreteness. It turns out that in a discrete space-time, we may be able to solve NP-hard problems in polynomial time.

1   Why Discrete Space-Time

Why discrete space. Traditional physics assumes that space and time are continuous. In most situations, this assumption works well, but a detailed analysis shows that in some cases, this continuity assumption leads to serious problems. One of such cases is the attempt to compute the overall energy of an electron – or of any other electrically charged elementary particle; see, e.g., [1, 11].

The overall energy of an electron can be computed as the sum of its "rest energy" – i.e., the energy $E_0 = m_0 \cdot c^2$ related to its rest mass m_0 – and the

overall energy E_{el} of its electric field. The electric field $\vec{E}(x)$ of an electron follow Coulomb's law according to which the value of this field at a point x at distance r from the electron is equal to $\dfrac{c}{r^2}$, for some constant c. It is known that the energy density $\rho(x)$ of an electric field is proportional to the square of the field itself, so $\rho(x) = \text{const} \cdot \left(\vec{E}(x)\right)^2$ and thus, $\rho(x) = \dfrac{c_1}{r^4}$ for some constant c_1.

An electron is an elementary particle, which means that it is not consisting of any interacting sub-particles. According to relativity theory, this implies that the electron is a single point in space. Indeed, if it would be spatially spread, it would have contained spatially different locations, then these locations would not be able to interact immediately – since, according to relativity theory, the speed of all communications is limited by the speed of light. That would mean that different parts of an electron act independently – and the main idea of an elementary particle is that this is not the case.

How that we know how the energy density $\rho(x)$ of the electric field depends on the spatial location, we can compute the overall energy of this field by integrating this density over the whole space: $E_{el} = \displaystyle\int \rho(x)\, dx = \int \dfrac{c_1}{r^4}\, dx$. In the polar coordinates, we could integrate over each sphere of radius r – which is equivalent to multiplying by the area $4\pi \cdot r^2$ of this sphere – and thus, get

$$E_{el} = \int\limits_0^\infty \frac{c_1}{r^4} \cdot 4\pi \cdot r^2 \, dr = c_1 \cdot 4\pi \cdot \int\limits_0^\infty \frac{1}{r^2} \, dr.$$

The resulting integral in the right-hand side is equal to the difference $\left.\dfrac{1}{r}\right|_0^\infty$. For $r = \infty$, this expression is 0, but for $r = 0$, it is infinite. So, we conclude that the overall energy of the electron's electric field is infinite and thus, that the overall energy of the electron is infinite – which, of course, makes no physical sense.

In this paradoxical result, we used pre-quantum physics. Pre-quantum physics has many similar paradoxes, e.g.:

- the black body radiation paradox according to which the overall energy emitted by the black body is infinite, or

- the paradoxical conclusion that atoms are impossible – since an accelerating electrically charged particle emits radiation and thus, electrons will eventually lose all their kinetic energy and fall on the nuclei.

These other paradoxes are, however, resolved in quantum physics – e.g., the black body radiation paradox was one of the main motivations for Max Planck to start quantum physics in the first place [1, 11].

Unfortunately, the above infinite-energy-of-the-electron paradox does not disappear if we take quantum effects into consideration; see, e.g., [1, 11]. Physicists have proposed several ways of solving this paradox. The mainstream approach to this paradox is to use the following procedure known as *renormalization*: crudely speaking, we assume that the rest mass of an electron is minus

infinity, so that the overall energy obtained by adding the rest energy and the infinite energy of the electric field remain finite.

In precise terms, in renormalization, instead of assuming that the distances can take any values from 0 to infinity, we artificially assume that distances cannot be smaller than a certain threshold $\varepsilon > 0$. If we integrate over such distances only, the integral

$$E_{el}(\varepsilon) = \int\limits_{\varepsilon}^{\infty} \frac{c_1}{r^4} \cdot 4\pi \cdot r^2 \, dr = c_1 \cdot 4\pi \cdot \int\limits_{\varepsilon}^{\infty} \frac{1}{r^2} \, dr \sim \frac{1}{r}\Big|_{\varepsilon}^{\infty} = \frac{1}{\varepsilon}$$

becomes finite. Thus, we can find the corresponding rest mass $m_0(\varepsilon)$ for which the overall energy $m_0(\varepsilon) \cdot c^2 + E_{el}(\varepsilon)$ is equal to the observed total energy of a stationary electron. As a result, for each $\varepsilon > 0$, we get physically meaningful finite values of different physical quantities. To find out what exactly the theory predicts, we then consider the limit values when $\varepsilon \to 0$.

To many physicists, the above procedure sounds more like a mathematical trick than a reasonable physical idea. Some of these physicists decided on a different approach:

- in their opinion, the above paradox shows that the usual assumption that the space-time is continuous (and that arbitrarily small distances are possible) leads to non-physical infinities;

- thus, it is reasonable to conclude that there is a lower bound on the distances between spatial points, i.e., the space is *discrete*.

From discrete space to discrete space-time. The above argument explains why the space proper is discrete. However, according to relativity theory, space and time are inter-related. What we consider a proper space (i.e., the set of all points considered at the same moment of time) depends on the frame of reference: points which look simultaneous to us will not be simultaneous anymore if we start moving.

Because of this inter-relation between space and time, once we consider discrete space, we should consider discrete time as well. Such discrete space-time models have indeed been seriously considered by physicists, including many famous one; see, e.g., [2, 3, 4, 8, 9, 10, 12, 13, 14, 15, 16] and references therein.

2   What We Can Compute in a Discrete Space-Time: Analysis of the Problem

Formulation of the problem. We more or less know what can be computed in the usual continuous space-time. It turns out that:

- some computational problems are tractable – in the sense that they can be efficiently solved in the usual space-time – and

- some are conjectured to be not tractable.

A natural question is: what can we compute if it turns out that space-time is actually discrete?

To answer this question, let us start by recalling the main concepts of feasibility and tractability related to computations in the usual (continuous) space-time. For detailed description and analysis of these concepts, see, e.g., [5, 7].

Feasible algorithms: a brief reminder. Whether an algorithm A is feasible or not depends on how fast its computation time t (i.e., in effect, the number of the corresponding elementary computational steps) grows with the length n of the input. When $t(n) = n$ or $t(n) = n^2$ or even $t(n) = n^3$, then for inputs of reasonable size the computation takes reasonable time; indeed:

- If we have an input of size 1 Kilobyte, i.e., if $n = 1000$, then for an algorithm that requires cubic time, we need 10^9 computational steps – which is less than a second on a modern computer.

- Even if $n = 10$ Kilobytes, we need about 10^3 seconds – less than an hour.

- For $n = 100$ Kilobytes, we may need a couple of weeks – or, better yet, a few hours on a highly parallel supercomputer.

On the other hand, if $t(n)$ grows exponentially with n, e.g., if $t(n) = 2^n$, then the computations become not realistic already for very reasonable input sizes. For example, for $n = 300$, the corresponding value $2^{300} \approx 10^{100}$ is larger than the number of particles in the Universe.

Because of these and similar examples, a current formal definition of feasibility is that $t(n)$ should be bounded by some polynomial of n; indeed:

- in the first two examples, $t(n) = n$, $t(n) = n^2$, and $t(n) = n^3$ are all polynomials, while

- it is known that the exponential function $t(n) = 2^n$ grows faster than any polynomial and thus, cannot be bounded by a polynomial.

Comment. The above formal definition is not perfect: e.g., a function $t(n) = 10^{100} \cdot n$ is a polynomial (and thus, feasible from the viewpoint of the formal definition), but it is clearly not practically feasible. However, this formal definition is the best we have.

Which problems are we solving. Based on the feasibility of algorithms, we can analyze tractability of problems. In computer applications, we consider well-defined problems, i.e., problems for which:

- once we have a candidate for a solution,

- we can check, in feasible time, whether this is indeed a solution.

Coming up with this candidate may be difficult, but once we found it, it should not be a problem to check. For example, it is often very difficult to prove a mathematical statement, but, once someone presents the proof in all details, even a computer can easily check it step-by-step without any difficulty. Such problems are known as *problems from the class NP*.

Comment. NP is an abbreviation of Non-deterministic Polynomial; here:

- polynomial means feasible (as in the formal definition), and

- non-deterministic computations means that, in addition to computational steps, we can make guesses.

In this sense, "non-deterministic polynomial" means that once we have guessed a solution, checking that this guess is indeed a solution requires feasible (polynomial) time – which is exactly what we mean by a well-defined problem.

NP-complete and NP-hard problems. Some problems from the class NP can be solved by feasible algorithms. Such problems are known as *tractable*. The class of all such problems is usually denoted by P (for Polynomial-time). Whether all problems from the class NP can be feasibly solved – i.e., whether the class NP simply coincides with its subclass P – this is still (2019) an open problem. Most computer scientists believe that P is different from NP, i.e., that some problems cannot be solved in feasible time – in other words, there are problems which are intractable.

What we do know is that there exists problems from the class NP which are as complex as possible – in the sense that every other problem from the class NP can be reduced to this problem. To be more precise, "reducing" a general problem A to a general problem B means that:

- given an instance a of the general problem A,

- we can feasibly compute an instance b of the general problem B

so that:

- once we have a solution to the instance b,

- we can feasibly transform it into a solution to the instance a (and every solution to a can be thus obtained).

Such problems are known as *NP-complete*.

NP-complete problems form a subclass of a more general class of complex problems – called *NP-hard* – to which also all the problems from the class NP can be reduced, but which do not necessarily belong to NP. For example, many optimization problems are NP-hard, but they are not necessarily in the class NP: if someone claims to have found an optimal solution, there is, in general, no easy way to check this claim rather than comparing this solution with all other alternatives – and this often takes exponential time.

To illustrate the notion of reduction, let us give a simple example. Suppose that we want to solve equations of the type $p \cdot x^2 + q \cdot \dfrac{1}{x^2} = r$. Solution of this equation can be reduced to solving a quadratic equation $a \cdot x^2 + b \cdot x + c = 0$. Indeed, if we multiply both sides of the original equation by x^2, introduce an auxiliary variable $y = x^2$, and move all the terms into the left-hand side, we get a quadratic equation $p \cdot y^2 - r \cdot y + q = 0$. Thus, from every instance (p, q, r) of the original problem, we can find the instance $(a, b, c) = (p, -r, q)$ of the quadratic-equations problem. Once we know the solutions y of the resulting quadratic equation, we can find the solutions x of the original equation by taking $x = \pm\sqrt{y}$.

There are many examples of NP-complete problems. In this paper, we will use the following example:

- given a list of positive integers $s_1, \ldots, s_n$,

- divide this list into two parts with the equal sums.

This problem is easy to describe in formal terms if we describe the division into two classes by introducing auxiliary variables $\varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\}$:

- we take $\varepsilon_i = 1$ if the i-th integer is assigned to the first part, and

- we take $\varepsilon_i = -1$ if the i-th integer is assigned to the second part.

In these terms, the desirable property is

$$\sum_{i:\varepsilon_i=1} s_i = \sum_{j:\varepsilon_j=-1} s_j,$$

i.e., equivalently,

$$\sum_{i=1}^{n} \varepsilon_i \cdot s_i = 0. \tag{1}$$

How to describe physics in discrete space-time? To analyze what can be computed in discrete space-time – i.e., by using the corresponding physical properties – it is important to understand how these physical processes can be described.

To answer this question, let us recall how physical processes are described in the usual continuous space-time; for details, see, e.g., [1, 11]. Traditionally, starting with Newton's mechanics, physical equations used to be described by differential equations. For example, the trajectory of a body affected by forces $F_i(t)$ depending on time can be described by Newton's second law $\dfrac{d^2 x_i}{dt^2} = F_i(t)$; electromagnetic field is described by Maxwell's equations, etc.

However, nowadays, new physical theories are no longer formulated in terms of differential equations. New theories are usually formulated in terms of the minimum action principle, according to which a quantity called *action S* is

minimized. Action S is equal to the integral over the whole space-time of an auxiliary function (called a *Lagrangian*) $L\left(f_i, \dfrac{\partial f_i}{\partial x_j}\right)$ whose value at each space-time point depends on the values, at this point, of the fields $f_i(x)$ and of their first derivatives $\dfrac{\partial f_i}{\partial x_j}$ with respect to all four space-time coordinates x_j:

$$S = \int L\left(f_i, \frac{\partial f_i}{\partial x_j}\right) d^4x.$$

The fact that many traditional physical equations can be described in this form was known since the early 19th century; it was used to solve some physical problems, and was, in general, viewed as one of the many alternative descriptions of a physical theory. However, with the appearance of quantum physics, it became clear that only theories which can be described in such minimum-action form are consistent with quantum ideas. Indeed, one of the ways to describe a quantum version of a physical theory is to use Feynman's *integration over trajectories* in which the corresponding complex-valued wave-function is proportional to the integral of the expression $\exp\left(\dfrac{\mathrm{i} \cdot S(\gamma)}{\hbar}\right)$ over all trajectories γ, where i denotes the imaginary unit $\sqrt{-1}$ and $\hbar$ is Planck's constant. When $\hbar \to 0$, this formula reduced to selecting the trajectory on which the action S attains its smallest possible value.

In discrete space-time:

- instead of the integral over the whole space-time – i.e., in effect, a weighted sum of the value of the Lagrangian at all the space-time points – it is natural to actually have the sum of the values of the corresponding Lagrangian over all space-time points;

- instead of the derivatives partial derivatives, i.e., expression of the type

$$\lim_{\Delta x_i \to 0} \frac{f(\ldots, x_i, \ldots) - f(\ldots, x_i - \Delta x_i, \ldots)}{\Delta x_i},$$

we should simply have finite differences

$$\frac{f(\ldots, x_i, \ldots) - f(\ldots, x_i - \Delta x_i, \ldots)}{\Delta x_i}$$

corresponding to the smallest possible values of Δx_i.

1-D case. In this paper, we consider Lagrangian corresponding to 1-D space. (This can be easily embedded into our 3-D space.) This means that instead of a continuum of possible value x of the spatial coordinate, we have discrete values

$$\ldots < x_0 < x_1 < \ldots < x_i < \ldots$$

Similarly, instead of a continuum of possible value t of time, we have discrete values

$$\ldots < t_0 < t_1 < \ldots < t_j < \ldots$$

A field $f(x,t)$ is thus described by its values $f(x_i,t_j)$ at all space-time points (x_i, t_j).

Thus, instead of the derivatives, we have differences of values between nearby points. So, instead of a spatial derivative, we have the ratios

$$\frac{f(x_i, t_j) - f(x_{i-1}, t_j)}{x_i - x_{i-1}}.$$

In other words, the value $L(x_i, t_j)$ of the Lagrangian at a space-time point (x_i, t_j) may depend not only on the values $f(x_i, t_j)$ of all the physical fields at this point, but also on the values of these fields at the neighboring spatial point (x_{i-1}, t_j).

Similarly, instead of a temporal derivative, we have the ratios

$$\frac{f(x_i, t_j) - f(x_i, t_{j-1})}{t_j - t_{j-1}}.$$

In other words, the value $L(x_i, t_j)$ of the Lagrangian at a space-time point (x_i, t_j) may depend not only on the values $f(x_i, t_j)$ of all the physical fields at this point and at the point (x_i, t_{j-1}), but also on the values of these fields at the previous moment of time (x_i, t_{j-1}).

Comment. From the purely mathematical viewpoint, we could start with a more standard definition of the temporal derivative as the limit

$$\lim_{\Delta t \to 0} \frac{f(x, t + \Delta t) - f(x, t)}{\Delta t}$$

and thus, get a somewhat different discrete version

$$\frac{f(x_i, t_{j+1}) - f(x_i, t_j)}{t_{j+1} - t_j}.$$

This would mean that the value of the Lagrangian $L(x_i, t_j)$ of the Lagrangian at a space-time point (x_i, t_j) may depend also on the values of these fields at the future moment of time (x_i, t_{j+1}). Mathematically, this makes perfect sense. However, from the physical viewpoint, dependence on the future moment of time violates our intuition about causality, according to which past events affect the future and not vice versa. So, to make our model more physically reasonable, we use the above equivalent definition of the derivative

$$\lim_{\Delta t \to 0} \frac{f(x, t) - f(x, t - \Delta t)}{\Delta t}.$$

This definition of a derivative may be somewhat less usual than the standard one, but it leads to a more physically acceptable description.

3 Main Result: How to Solve NP-Hard Problems in Polynomial Time

Main idea. We will provide an example of a reasonable discrete-space-time Lagrangian – in the sense described in the previous section – for which the corresponding minimization is NP-hard. Thus, if we can implement this Lagrangian, then, by setting up proper initial conditions and observing the future state of the system, we will get a solution to an NP-hard problem.

By definition, as we have mentioned, a problem is NP-hard if every other problem from the class NP can be reduced to this problem. So, if have a polynomial-time algorithm for solving one NP-hard problem, then, by applying the corresponding reduction, we will be able to solve *all* problems from the class NP in polynomial (i.e., feasible) time.

Let us now write down the desired Lagrangian.

The Lagrangian: the formula. Let us consider the case when we have three fields: $f(x,t)$, $\varepsilon(x,t)$, and $z(x,t)$. Let us consider the following Lagrangian:

$$L(x_i, t_j) = [f(x_i, t_j) - (f(x_{i-1}, t_j) + \varepsilon(x_i, t_j) \cdot f(x_i, t_{j-1}))]^2 +$$

$$\left[\varepsilon^2(x_i, t_j) - 1\right]^2 + [z(x_i, t_{j-1}) \cdot f(x_i, t_j)]^2. \tag{2}$$

In other words, nature minimizes the following action:

$$S = \sum_{i,j} L(x_i, t_j). \tag{3}$$

How to use this Lagrangian to solve the corresponding NP-hard problem. Let us assume that we are given an instance of the above problem (1), i.e., we are given n positive integers $s_1, \ldots, s_n$. We want to find the values $\varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\}$ that satisfy the equation (1).

To find these values ε_i, at the current moment of time t_0, we set the following initial conditions:

- we set $f(x_i, t_0) = s_i$ for $i = 1, \ldots, n$ and $f(x_i, t_0) = 0$ for all other i;

- we also set $z(x_0, t_0) = z(x_n, t_0) = 1$ and $z(x_i, t_0) = 0$ for all other i.

The values of the fields at the next moment of time t_1 will then be determined as the values that minimize the expression (3).

Let us show that if the original instance of the NP-hand problem (1) has a solution, then this solution will appear as values $\varepsilon_i = \varepsilon(x_i, t_1)$. In other words, we will have

$$\sum_{i=1}^{n} \varepsilon(x_i, t_1) \cdot s_i = 0. \tag{4}$$

Indeed, if the current instance of the problem (1) has a solution $\varepsilon_1, \ldots, \varepsilon_n$, then we can take $\varepsilon(x_i, t_1) = \varepsilon_i$ and $f(x_i, t_1) = \sum_{k=1}^{i} \varepsilon_k \cdot s_k$. In this case,

- for all i, we have

$$f(x_i, t_1) = \sum_{k=1}^{i} \varepsilon_k \cdot s_k =$$

$$\left(\sum_{k=1}^{i-1} \varepsilon_k \cdot s_k \right) + \varepsilon_i \cdot s_i = f(x_{i-1}, t_1) + \varepsilon(x_i, t_1) \cdot f(x_i, t_0);$$

  thus, the first term in the expression (2) is equal to 0;

- for all i, we have $\varepsilon(x_i, t_1) = \pm 1$, hence $\varepsilon^2(x_i, t_1) = 1$ and therefore, the second term in the expression (2) is equal to 0;

- for all $i \neq 0$ and $i \neq n$, we have $z(x_i, t_0) = 0$ hence $z(x_i, t_0) \cdot f(x_i, t_1) = 0$;

- for $i = 0$, we have $f(x_0, t_1) = 0$ thus also $z(x_0, t_0) \cdot f(x_0, t_1) = 0$;

- for $i = n$, we have $f(x_n, t_1) = \sum_{k=1}^{n} \varepsilon_k \cdot s_k = 0$ (since the values ε_i solve the problem (1)), thus also $z(x_n, t_0) \cdot f(x_n, t_1) = 0$; so, the third term in the Lagrangian is also equal to 0.

Thus, the Lagrangian is equal to 0, so the action is equal to 0 – and since the action is the sum of squares, 0 is its smallest possible value. So, if the given instance of the original NP-hard problem has a solution, the action can reach its smallest possible value 0.

Let us prove that, vice versa, if the action reaches 0, this means that the values $\varepsilon(x_i, t_1)$ form the solution to the given instance of the original problem (1). Indeed, the sum of squares is 0, this means that all the squares are equal to 0 and hence, each of the squared terms in the expression (2) is equal to 0.

In particular, from the fact that $z(x_0, t_0) \cdot f(x_0, t_1) = 0$, taking into account that $z(x_0, t_0) = 1$, we conclude that $f(x_0, t_1) = 0$. Now, for each i, from the equality

$$f(x_i, t_1) - (f(x_{i-1}, t_1) + \varepsilon(x_i, t_1) \cdot f(x_i, t_0)) = 0,$$

taking into account that $f(x_i, t_0) = s_i$, we conclude that $f(x_i, t_1) = f(x_{i-1}, t_1) + \varepsilon(x_i, t_1) \cdot s_i$. Since for $i = 0$, we have $f(x_0, t_1) = 0$, we can thus prove, by induction, that for all i, we have $f(x_i, t_1) = \sum_{k=1}^{i} \varepsilon(x_k, t_1) \cdot s_k$. In particular, for $i = n$, we get

$$f(x_n, t_1) = \sum_{k=1}^{n} \varepsilon(x_k, t_1) \cdot s_k.$$

From the fact that $z(x_n, t_0) \cdot f(x_n, t_1) = 0$, taking into account that $z(x_n, t_0) = 1$, we conclude that $f(x_n, t_1) = 0$, i.e., that

$$\sum_{k=1}^{n} \varepsilon(x_k, t_1) \cdot s_k = 0. \tag{5}$$

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From the equality $\varepsilon^2(x_i, t_1) - 1 = 0$, we conclude that $\varepsilon^2(x_i, t_1) = 1$ hence $\varepsilon(x_i, t_1) \in \{-1, 1\}$ and thus, that the values $\varepsilon_i = \varepsilon(x_i, t_1)$ indeed solve the original instance of the above NP-hard problem.

The statement is proven.

Comment. Discretization of space-time can be viewed as particular example of quantization – indeed, quantum mechanics started by physicists realizing that electromagnetic waves of a given frequency (e.g., visible light) cannot have an arbitrary small energy: there is the smallest amount of such energy (quantum), and any other amount is proportional to this one. Quantum physics continued with observing that an atom cannot be in a state with any possible energy: there are discrete energies, etc.

From this viewpoint, our result about computing in a discrete space-time can be viewed as a particular case of – broadly understood – quantum computing; see, e.g., [6]

Acknowledgments

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