If Space-Time Is Discrete, It Could Be Possible to Solve NP-Complete Problems in Polynomial Time

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Traditional physics assumes that space and time are continuous. However, a deeper analysis shows that this seemingly reasonable space-time model leads to some serious physical problems. One of the approaches that physicists have proposed to solve these problems is to assume that the space-time is actually discrete. In this paper, we analyze possible computational consequences of this discreteness. It turns out that in a discrete space-time, it is probably possible to solve NP-complete problems in polynomial time: namely, this is possible in almost all physically reasonable models of dynamics in discrete space-time (almost all in some reasonable sense).

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1 FORMULATION OF THE PROBLEM

Need for faster computations. While modern computers are very fast – several orders to magnitude faster than computers were a few decades ago – there are still practical problems for which it is desirable to have even faster computations. For example, in principle, we can spend several hours on a high performance computer and find out, with high probability, where the tornado will move in the next 15 minutes – just like we can usually predict, with high probability, the next day's weather. However, while it is OK to spend several hours to predict tomorrow's weather, for tornados, the same computation time makes no sense: we get the computation results only after the tornado already moved. This – and other – problems show that there is still a need to drastically speed up computations.

NP-completeness: fundamental obstacle to speeding up computations. So far, engineering and algorithmic progress have been making computations faster and faster. However, there is a fundamental theoretical limit to what we can achieve within the existing physics. This limit is related to the notion of NP-completeness (for readers who are not very familiar with this notion, we provide a detailed explanation in Section 2). Crudely speaking, provided that P is different from NP (which most computer scientists believe to be true), it means that there are some problems that require un-realistic exponential time to be solved.

Since we cannot solve these problems by using the currently used physical processes and models, many researchers have been checking whether, by using other possible physical processes and models, we will be able to solve NP-hard problems in polynomial time.

What we do in this paper. In this paper, we provide a new example of such possibility: namely, we show that in discrete space-time, almost all (in some reasonable sense) dynamical equations lead us to the possibility to solve NP-complete problems in polynomial time.

Comments.

- The very fact that there are space-time models in which we can solve NP-complete problems in feasible (polynomial) time is well known. For example, it is known that in the simplest example of curved space – Lobachevsky space – the volume of a ball of radius R grows exponentially with R; see, e.g., [1]. Thus, we can place exponentially many processors within this ball – and hence, perform exponentially many

computational steps in parallel; see, e.g., [14, 15, 16, 17]. In this paper, we provide another example of physical space-time in which this is possible.

- It should be mentioned that discreteness of physical space-time is essential for this result: the proposed procedure does not work in continuous space-time models. At present, there are some arguments why physical space-time may be discrete – we present these arguments later in this paper – but there is no specific information about this discreteness and thus, we cannot yet implement our scheme. Hopefully, in the future, if and when space-time will turn out to be discrete, this scheme will be applicable.

- Finally, we should emphasize that we are *not* saying that P = NP; the statement P = NP – that we will describe later in precise terms – is a precisely defined mathematical statement, we do not know whether it is true or not, and our results do not bring us closer to understanding whether this statement is true or not. What we *are* claiming is that even if the above mathematical statement is false, and P is different from NP – which means that a usual Turing machine cannot solve all NP-problems in polynomial time – in some special models of space-time (namely, in discrete space-time models), we will be able to set up a computational device that will solve these problems in polynomial time.

Intended audience. To fully understand our result, a reader must be aware of the basic ideas from computer science – such as NP-completeness and related notions – and of the basic ideas of physics – to understand the corresponding physical models.

To make this result fully accessible both to computer scientists and to physicists, we therefore added brief explanations of both classes of notions. We hope that, as a result, this paper will be understandable both to computer scientists and to physicists. Of course, computer science readers who are familiar with NP-related notions can skip the corresponding explanations, and, similarly, physicist readers will only need to glance through our general physics-related introductions.

The structure of this paper. Our intent to make our paper understandable both to computer scientists and to physicists motivated the following paper's structure. Section 2 briefly explains what is P, what is NP, and what it means

for a problem to be NP-complete. Readers who are already familiar with these notions can easily skip this section. Section 3 briefly overviews some previous attempts to come up with possible physical processes potentially allowing us to speed up computations; for these processes, we explain both the advantages and the limitations of these proposed schemes.

In Section 4, we explain why many physicists believe that space-time may be discrete. Expanding on this, in Section 5, we provide a general explanation of possible dynamical equations in discrete space-time. Finally, in Section 6, we use this general explanation to prove that in almost all such models, it is possible to solve NP-complete problems in polynomial time.

2 P, NP, NP-COMPLETE, ETC.: A BRIEF REMINDER

Why we need this section. The notions of P, NP, and NP-completeness are familiar to most computer scientists. However, since, as have mentioned earlier, we want this paper to be understandable both to computer scientists and to physicists, we have added this section, with a brief explanation of these notions.

Readers already familiar with these notions can skip this section.

What we do in this section. In this section, we briefly overview the main concepts of feasibility and tractability related to computations. For detailed description and analysis of these concepts, see, e.g., [13, 19].

Feasible algorithms: a brief reminder. Some algorithms are *practically feasible*, while other algorithms require so many computational steps even for reasonable-size inputs that even on the fastest computers, computations would take longer than the lifetime of the Universe. Whether an algorithm A is practically feasible or not depends on how fast its computation time t (i.e., in effect, the number of the corresponding elementary computational steps) grows with the length n of the input. When $t(n) = n$ or $t(n) = n^2$ or even $t(n) = n^3$, then for inputs of reasonable size the computation takes reasonable time; indeed:

- If we have an input of size 1 Kilobyte, i.e., if $n = 1000$, then for an algorithm that requires cubic time, we need 10^9 computational steps – which is less than a second on a modern computer.
- Even if $n = 10$ Kilobytes, we need about 10^3 seconds – less than an hour.

- For $n = 100$ Kilobytes, we may need a couple of weeks – or, better yet, a few hours on a highly parallel supercomputer.

On the other hand, if $t(n)$ grows exponentially with n, e.g., if $t(n) = 2^n$, then the computations become not realistic already for very reasonable input sizes. For example, for $n = 300$, the corresponding value $2^{300} \approx 10^{100}$ is larger than the number of particles in the Universe.

Motivated by these and similar examples, a current *formal* definition of feasibility is that $t(n)$ should be bounded by some polynomial of n; indeed:

- in the first two examples, $t(n) = n$, $t(n) = n^2$, and $t(n) = n^3$ are all polynomials, while
- it is known that the exponential function $t(n) = 2^n$ grows faster than any polynomial and thus, cannot be bounded by a polynomial.

Comment. It is important to mention that for most algorithms, the above formal definition adequately describes what is practically feasible and what is not – but not always. For example, a function $t(n) = 10^{100}n$ is a polynomial (and thus, feasible from the viewpoint of the formal definition), but it is clearly not practically feasible. On the other hand, a function $t(n) = \exp(10^{-80}n)$ grows faster than any polynomial – as is, thus, not feasible from the viewpoint of the formal definition – but its values are quite reasonable even for very large n, so an algorithm with this time complexity is clearer practically feasible.

Which problems are we solving. Based on the feasibility of algorithms, we can analyze tractability of problems. In computer applications, we consider well-defined problems, i.e., problems for which:

- once we have a candidate for a solution,
- we can check, in feasible time, whether this is indeed a solution.

Coming up with this candidate may be difficult, but once we found it, it should not be a problem to check. For example, it is often very difficult to prove a mathematical statement, but, once someone presents a proof in all the details, even a computer can easily check it step-by-step without any difficulty. Such problems are known as *problems from the class NP*.

Comment. NP is an abbreviation of Non-deterministic Polynomial; here:

- polynomial means feasible (as in the formal definition), and

- non-deterministic computations means that, in addition to computational steps, we can make guesses.

In this sense, "non-deterministic polynomial" means that once we have guessed a solution, checking that this guess is indeed a solution requires feasible (polynomial) time – which is exactly what we mean by a well-defined problem.

NP-complete and NP-hard problems. Some problems from the class NP can be solved by feasible algorithms. Such problems are known as *tractable*. The class of all such problems is usually denoted by P (for Polynomial-time). Whether all the problems from the class NP can be feasibly solved – i.e., whether the class NP simply coincides with its subclass P – this is still (2020) an open problem. Most computer scientists believe that P is different from NP, i.e., that some problems cannot be solved in feasible time – in other words, that there are problems which are intractable.

What we *do* know is that there exist problems from the class NP which are as complex as possible – in the sense that every other problem from the class NP can be reduced to this problem. To be more precise, "reducing" a general problem A to a general problem B means that:

- given an instance a of the general problem A,
- we can feasibly compute an instance b of the general problem B

so that:

- once we have a solution to the instance b,
- we can feasibly transform it into a solution to the instance a (and every solution to a can be thus obtained).

Such problems are known as *NP-complete*.

NP-complete problems form a subclass of a more general class of complex problems – called *NP-hard* – to which also all the problems from the class NP can be reduced, but which do not necessarily belong to NP. For example, many optimization problems are NP-hard, but they are not necessarily in the class NP: if someone claims to have found an optimal solution, there is, in general, no easy way to check this claim rather than comparing this solution with all the other alternatives – and this often takes exponential time.

What we mean by reduction: a clarifying example. The above definitions of NP-completeness and NP-hardness were based on the notion of reduction. To clarify this notion, let us give a simple example.

Suppose that we want to solve equations of the type $pz^2 + \dfrac{q}{z^2} = m$, in which p, q, and m are known and we need to find the unknown z. Solution of this equation can be reduced to solving a quadratic equation $ay^2 + by + c = 0$. Indeed, if we multiply both sides of the original equation by z^2, introduce an auxiliary variable $y = z^2$, and move all the terms into the left-hand side, we get a quadratic equation $py^2 - my + q = 0$. Thus, from every instance (p, q, m) of the original problem, we can find the instance $(a, b, c) = (p, -m, q)$ of the quadratic-equations problem. Once we know the solutions y of the resulting quadratic equation, we can find the solutions z of the original equation by taking $z = \pm\sqrt{y}$.

An example of an NP-complete problem. There are many examples of NP-complete problems. In this paper, we will use the following known example:

- given a list of positive integers $s_1, \ldots, s_n$,
- divide this list into two parts with the equal sums.

This problem is easy to describe in formal terms if we describe the division into two classes by introducing auxiliary variables $\varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\}$:

- we take $\varepsilon_i = 1$ if the i-th integer is assigned to the first part, and
- we take $\varepsilon_i = -1$ if the i-th integer is assigned to the second part.

In these terms, the desirable property is

$$\sum_{i:\varepsilon_i=1} s_i = \sum_{j:\varepsilon_j=-1} s_j,$$

i.e., equivalently,

$$\sum_{i=1}^{n} \varepsilon_i s_i = 0. \tag{1}$$

Comment. To avoid confusion, we would like to again emphasize that we are *not* claiming that P = NP, this question is a formal hypothesis not depending on any physics. What we do show is that even if P $\neq$ NP – so that on normal computers, we will still need non-feasible time to solve NP-complete problems – within appropriate discrete-time models, we will potentially be able to solve NP-complete problems in polynomial time.

3 PHYSICAL PROCESSES PREVIOUSLY PROPOSED FOR SOLVING NP-COMPLETE PROBLEM IN POLYNOMIAL TIME: A BRIEF OVERVIEW

What is known. It is known that even in the usual (continuous) space-time, it is possible to come up with possible dynamical equations such that if there is a physical process changing according to these equations, then it will be possible to solve NP-complete problems in polynomial time. Moreover, with some equations, it is even possible to solve problems which are not algorithmically solvable on the usual computers at all – e.g., the problem of checking whether a given program will eventually stop or it will continue indefinitely. Readers interested in such equations can look at the papers [3, 4, 5, 6, 7] and at references therein.

Comment. One should not be surprised that such models exist. Indeed, in a computer, every program is represented as a long sequence of 0s and 1s. We can encode 0, e.g., by spin down, and 1 by spin up. Then, in principle, one can have the following physical model for checking whether a program will halt: in this model, n linearly placed particles with spins down or up will attract each other if and only if the corresponding sequence of 0s and 1s corresponds to a syntactically correct Java program that halts. Then, we can check whether a given program – i.e., a given sequence of 0s and 1s – halts by placing particles with appropriately oriented spins in a line and checking whether they attract each other.

Of course, the models described in [3, 4, 5, 6, 7] are more physical – an thus, more interesting – that the above simple idea.

Limitations of the previously proposed schemes. While the previously proposed physical models are much more physical than the above simple ideas, it is still not clear how to implement them – and thus, it is not clear whether the corresponding physical processes exist in the real world.

What we show in this paper. In this paper, we show that the above limitations are caused by the fact that the previous authors considered *continuous* space-time models. It turns out that in *discrete* space-time, polynomial solution of NP-complete problems is possible in almost all models – and we also explain why many physicists believe that the actual space-time is indeed discrete.

4 WHY DISCRETE SPACE-TIME

Why discrete space. Traditional physics assumes that space and time are continuous. In most situations, this assumption works well, but a detailed analysis shows that in some cases, this continuity assumption leads to serious problems. One of such cases is an attempt to compute the overall energy of an electron – or of any other electrically charged elementary particle; see, e.g., [8, 24].

The overall energy of an electron can be computed as the sum of its "rest energy" – i.e., the energy $E_0 = m_0 c^2$ related to its rest mass m_0 – and the overall energy E_{el} of its electric field. The electric field $\vec{E}(\vec{x})$ of an electron follows Coulomb's law according to which the value of this field at a point $\vec{x}$ at distance r from the electron is equal to $\dfrac{c}{r^2}$, for some constant c. It is known that the energy density $\rho(\vec{x})$ of an electric field is proportional to the square of value of the field itself, so $\rho(\vec{x}) = \text{const} \left(\vec{E}(\vec{x})\right)^2$ and thus, $\rho(\vec{x}) = \dfrac{c_1}{r^4}$ for some constant c_1.

An electron is an elementary particle, which means that it is not consisting of any interacting sub-particles. According to relativity theory, this implies that the electron is a single point in space. Indeed, if it would be spatially spread, it would have contained spatially different locations, then these locations would not be able to interact immediately – since, according to relativity theory, the speed of all communications is limited by the speed of light. That would mean that different parts of an electron act independently – and the main idea of an elementary particle is that this is not the case.

Now that we know how the energy density $\rho(\vec{x})$ of the electric field depends on the spatial location, we can compute the overall energy of this field by integrating this density over the whole space:

$$E_{el} = \int \rho(\vec{x})\, d\vec{x} = \int \frac{c_1}{r^4}\, d\vec{x}.$$

This integral is easy to compute if we use polar coordinates centered in the electron. In these polar coordinates, we could integrate over each sphere of radius r – which is equivalent to multiplying by the area $4\pi r^2$ of this sphere – and thus, get

$$E_{el} = \int_0^\infty \frac{c_1}{r^4}\, 4\pi r^2\, dr = 4\pi c_1 \int_0^\infty \frac{1}{r^2}\, dr.$$

The resulting integral in the right-hand side is proportional to the difference

$\left.\frac{1}{r}\right|_0^\infty$. For $r = \infty$, this expression is 0, but for $r = 0$, it is infinite. So, we conclude that the overall energy of the electron's electric field is infinite and thus, that the overall energy of the electron is infinite – which, of course, makes no physical sense.

In this paradoxical result, we used pre-quantum physics. Pre-quantum physics has many similar paradoxes, e.g.:

- the black body radiation paradox, according to which the overall energy emitted by the black body is infinite, or
- the paradoxical conclusion that atoms are impossible – since an accelerating electrically charged particle emits radiation and thus, the electrons will eventually lose all their kinetic energy and fall on the nuclei.

These other paradoxes are, however, resolved in quantum physics – e.g., the black body radiation paradox was one of the main motivations for Max Planck to start quantum physics in the first place [8, 24].

Unfortunately, the above infinite-energy-of-the-electron paradox does not disappear if we take quantum effects into consideration; see, e.g., [8, 24]. Physicists have proposed several ways of solving this paradox. The mainstream approach to this paradox is to use the following procedure known as *renormalization*: crudely speaking, we assume that the rest mass of an electron is minus infinity, so that the overall energy obtained by adding the rest energy and the infinite energy of the electric field remain finite.

In precise terms, in renormalization, instead of assuming that the distances can take any values from 0 to infinity, we artificially assume that distances cannot be smaller than a certain threshold $\varepsilon > 0$. If we integrate over such distances only, the integral

$$E_{el}^{\varepsilon} = \int_{\varepsilon}^{\infty} \frac{c_1}{r^4}\, 4\pi r^2\, dr = c_1 4\pi \int_{\varepsilon}^{\infty} \frac{1}{r^2}\, dr \sim \left.\frac{1}{r}\right|_{\varepsilon}^{\infty} = \frac{1}{\varepsilon}$$

becomes finite. Thus, we can find the corresponding rest mass m_0^{ε} for which the overall energy $m_0^{\varepsilon} c^2 + E_{el}^{\varepsilon}$ is equal to the observed total energy of a stationary electron. As a result, for each $\varepsilon > 0$, we get physically meaningful finite values of different physical quantities. To find out what exactly the theory predicts, we then consider the limit values when $\varepsilon \to 0$.

To many physicists, the above procedure sounds more like a mathematical trick than a reasonable physical idea. Some of these physicists decided on a different approach:

- in their opinion, the above paradox shows that the usual assumption that the space-time is continuous (and that arbitrarily small distances are possible) leads to non-physical infinities;
- thus, it is reasonable to conclude that there is a lower bound on the distances between spatial points, i.e., that the space is *discrete*.

From discrete space to discrete space-time. The above argument explains why the space proper is discrete. However, according to relativity theory, space and time are inter-related. What we consider a proper space (i.e., the set of all points considered at the same moment of time) depends on the frame of reference: points which look simultaneous to us will not be simultaneous anymore if we start moving.

Because of this inter-relation between space and time, once we consider discrete space, we should consider discrete time as well. Such discrete space-time models have indeed been seriously considered by physicists, including many famous ones; see, e.g., [2, 9, 10, 11, 12, 20, 21, 22, 23, 25, 26, 27, 28, 29] and references therein.

Comment. While the above arguments in favor of discreteness of proper space are reasonably convincing, arguments in favor of discrete time are not that strong. So, in principle, it is possible to consider models with discrete proper space and continuous time – and several physicists have indeed considered such models. From the purely physical viewpoint, such mixed discrete-continuous models make perfect sense. However, what we are interested in is computability. Let us explain that from this viewpoint, mixed space-time models do not add much to our ability to compute, and we indeed need to consider discrete time in addition to discrete space.

Indeed, in general, the state of the world at any given moment of time t is described by listing, for each spatial location $\vec{x}$, the values of all the physical quantities $q_i(\vec{x}, t)$ at this location at the given moment of time. Physical equations describe how this state changes with time. For continuous space and continuous time, this means that the state of the world is described by a function of three spatial variables. Thus, to describe how this state changes, we need to describe how the corresponding rate of change $\dfrac{\partial q_i}{\partial t}$ depends on the state itself:

$$\frac{\partial q_i(\vec{x}, t)}{\partial t} = F(q_1(., t), q_2(., t), \ldots)$$

for some functional $F(q_1, q_2, \ldots)$. This change is usually described by a partial differential equation, in which case $F(q_1(., t), q_2(., t), \ldots)$ depends

only on the values $q_1(\vec{x}, t), q_2(\vec{x}, t), \ldots$ and of their spatial derivatives at the same spatial point $\vec{x}$.

Such equations may be sometimes cumbersome to solve, but in principle, they can all be approximately solved by discretizing time (which corresponds to Euler's method of solving differential equations):

- we start with the state at moment t;
- based on this state, we compute the state at moment $t + \Delta t$ as

$$q_i(\vec{x}, t + \Delta t) \approx q_i(\vec{x}, t) + F(q_1(., t), q_2(., t) \ldots,) \, \Delta t,$$

  for some small Δt,
- then we compute the state at moment $t + 2\Delta t$ based on the state at moment $t + \Delta t$, etc.,

until we reach the final moment T. This is feasible and thus, we do not expect that it will lead to solving NP-complete problems.

When space is discrete and time is continuous, the state of the world is described by the values $q_{ij} \stackrel{\text{def}}{=} q_i(\vec{x}_j)$ corresponding to all possible spatial points $\vec{x}_j$. In this case, instead of the partial differential equation, we have a system of ordinary differential equations describing the derivatives $\dfrac{dq_{ij}}{dt}$. Such equations can also be solved by the feasible Euler's method – and thus, we do not expect them to solve NP-complete problems.

The case of discrete time is different, since in this case, we cannot directly reduce the dynamics to small easy-to-compute steps of going from moment t to a very close moment of time $t + \Delta t$. Of course, this does not necessarily mean that the corresponding dynamical problems are NP-complete, this requires a deeper mathematical analysis of the problem, an analysis that we provide in this paper.

5 POSSIBLE PHYSICAL EQUATIONS IN DISCRETE SPACE-TIME

We need to understand the physics of discrete space-time. To analyze what can be computed in discrete space-time – i.e., by using the corresponding physical properties – it is important to understand how these physical processes can be described.

To answer this question, let us first recall how physical processes are described in the usual continuous space-time; for details, see, e.g., [8, 24].

Usual case of continuous space-time: how physical theories are described? In the past, starting with Newton's mechanics, physical processes used to be described by differential equations. For example:

- the trajectory of a body affected by forces $F_i(t)$ (which are, in general, depending on time) can be described by the Newton's second law

$$\frac{d^2 x_i}{dt^2} = F_i(t);$$

- the electromagnetic field is described by the Maxwell's equations, etc.

However, nowadays, new physical theories are no longer formulated in terms of differential equations. New theories are usually formulated in terms of the minimum action principle, according to which a quantity called *action* S is minimized. The action S is equal to the integral over the whole space-time of an auxiliary function (called a *Lagrangian*) $L\left(f_i, \dfrac{\partial f_i}{\partial x_j}\right)$ whose value at each space-time point depends on the values, at this point, of the fields $f_i(x)$ and of their first derivatives $\dfrac{\partial f_i}{\partial x_j}$ with respect to all four space-time coordinates x_j:

$$S = \int L\left(f_i, \frac{\partial f_i}{\partial x_j}\right)\, d^4 x.$$

The fact that traditional physical equations – Newton's mechanics, Maxwell's equations, etc. – can be described in this form was known since the early 19th century; this fact was used to solve some physical problems, and was, in general, viewed as one of the many alternative descriptions of a physical theory.

However, with the appearance of quantum physics, it became clear that this is not simply one of the possible ways to formulate a theory: it turned out that only theories which can be described in such minimum-action form are consistent with quantum ideas. Indeed, one of the ways to describe a quantum version of a physical theory is to use Feynman's *integration over trajectories* in which the corresponding complex-valued wave-function is proportional to the integral of the expression $\exp\left(\dfrac{\mathrm{i}S(\gamma)}{\hbar}\right)$ over all trajectories γ, where i denotes the imaginary unit $\sqrt{-1}$ and $\hbar$ is Planck's constant. When $\hbar \to 0$, this formula reduced to selecting the trajectory on which the action S attains its smallest possible value.

In terms of action, the dynamics of a physical system works as follows:

- we have (or we set up) the initial conditions of all the corresponding fields,
- then, nature automatically sets up future values of all these fields in such a way as to minimize the value of the corresponding action S.

Discrete space-time analogues of integration and differentiation. How can we apply these ideas to the case of discrete space-time?

In the continuous space-time, the action functional S – i.e., the quantity whose minimization described the dynamics of all the physical fields – is described as the integral of the Lagrange function L over time and over all spatial coordinates. The value of the Lagrangian L at any point in space-time depends on the values of all the fields and of their first derivatives at this same space-time point. What are the discrete space-time analogues of these two notions?

To get natural space-time analogues of integration and differentiation, let us recall that, from the mathematical viewpoint:

- the integral $\int f(x)\,dx$ is defined as a limit of the weighted sums $\sum f(x_i)\,\Delta x$ when the distance Δx between the points x_i tends to 0, and
- the derivative $\dfrac{df}{dx}$ is defined as the limit of the ratios $\dfrac{f(x+\Delta x)-f(x)}{\Delta x}$ when Δx tends to 0.

From the physical viewpoint, the possibility of having arbitrary small Δx is often an idealization. For example, when we use partial derivatives in elasticity theory, we realize that the corresponding ratio makes physical sense only for the values Δx which are much larger than the distance between the atoms:

- when we get to smaller distances Δx,
- we can no longer assume that the solid body is homogeneous, we need to take into account the corresponding crystal structure.

So, from the physical viewpoint, even when explicitly write a derivative, what we really mean is the ratio corresponding to some small but finite value Δx.

Based on the logic, in discrete space-time, the action is simply a weighted sum $\sum w_{ij}\,L(x_i,t_j)$ of the values of the Lagrangian corresponding to different points in space-time. We can simplify this description even further if, instead of the original Lagrangian, we consider the product $w_{ij}\,L(x_i,t_j)$ as the

new Lagrangian. In this description, the action is simply the sum $\sum L(x_i, t_j)$ of the Lagrangian values corresponding to different space-time points.

Similarly, a natural discrete space-time analogue of the spatial derivative is the ratio $\dfrac{f(x_{i+1}, t_j) - f(x_i, t_j)}{x_{i+1} - x_i}$.

With time derivative, the situation is somewhat more complicated. At first glance, it may seem reasonable to consider, as the value of the time derivatibe at the space-time point (x_i, t_j), a similar expression

$$\frac{f(x, t + \Delta t) - f(x, t)}{\Delta t} = \frac{f(x_i, t_{j+1}) - f(x_i, t_j)}{t_{j+1} - t_j}.$$

However, this would mean that the value $L(x_i, t_j)$ of the Lagrangian at a space-time point (x_i, t_j) may depend also on the values of these fields at the *future* moment of time (x_i, t_{j+1}). Mathematically, this may make perfect sense, but from the physical viewpoint, dependence on the future moments of time violates our intuition about causality, according to which past events affect the future and not vice versa. So, to make our description physically reasonable, we need to use a mathematically equivalent definition of the derivative as

$$\lim_{\Delta t \to 0} \frac{f(x, t) - f(x, t - \Delta t)}{\Delta t}.$$

In line with this definition, as natural discrete space-time analogue of time derivative, we will use the ratio $\dfrac{f(x_i, t_j) - f(x_i, t_{j-1})}{t_j - t_{j-1}}$.

The resulting description of a general physical theory in discrete space-time. We have a finite set of *fields*, i.e., functions $f_1(x, t)$, ..., $f_k(x, t)$ that assign a numerical value $f_a(x, t)$ to each discrete space-time point (x, t). By a *spatial derivative* $D_x f_a(x, t)$, we mean a ratio $\dfrac{f_a(x', t) - f_a(x, t)}{d(x, x')}$, where x' is a point in a close vicinity of x. Similarly, by a *temporal* derivative $D_t f_a(x_i, t_j)$, we mean a ratio $\dfrac{f_a(x_i, t_j) - f_a(x_i, t_{j-1})}{t_j - t_{j-1}}$. A *Lagrangian* is a function $L(f_1, \ldots, f_k, D_x f_1, \ldots, D_x f_k, D_t f_1, \ldots, D_t f_k)$ whose value $L(x, t)$ at each space-time point (x, t) depends on the values of all the fields and their spatial and temporal derivatives at this points (x, t).

Once we know the values of all the fields at some moment t_0, the values at the following moments of time are determined by the condition that the *action* – i.e., the sum $S = \sum\limits_i \sum\limits_j L(x_i, t_j)$ – attains its smallest possible value.

6 MAIN RESULT: FOR ALMOST ALL PHYSICAL THEORIES IN DISCRETE SPACE-TIME, IT IS POSSIBLE TO SOLVE NP-COMPLETE PROBLEMS IN POLYNOMIAL TIME

Now, we are ready to formulate and prove our main result – that for almost all physical theories in discrete space-time, it is possible to solve NP-complete problems in polynomial time.

6.1 Towards Formulating the Problem in Precise Terms

What do we mean by "almost all". In mathematics, the words "almost all" have several meanings.

The most widely used meaning is that we have a probability measure, and "almost all" means that a given property is satisfied with probability 1. This definition will depend on the specific choice of the probability measure: what has probability 1 in one measure may have probability less than 1 (or even 0) in another measure. So, to specify the meaning, we need to select a probability measure. In some physical situation – e.g., in statistical physics – there is a natural probability measure on the set of all possible configurations.

However, in our case, we are interested in interpreting the term "almost all physical theories". On the set of all physical theories, there does not seem to be a natural probability measure.

What *can* be naturally defined on the set of all possible theories is a natural notion of closeness – i.e., in mathematical terms, natural *metric* (distance) and related topology. The main idea is straightforward: if two Lagrangians have similar values for all reasonable values of fields, then it is natural to consider these two Lagrangians to be close to each other. To be more precise, we can say that two Lagrangians L and L' are ε-close, for some small $\varepsilon > 0$, if for some fixed large number $\Delta > 0$, if the values of all the fields do not exceed Δ, then the values of the Lagrangians L and L' differ by no more than ε.

In situations when such closeness can be naturally defined, there is another notion of "almost all" – a distance-based one: we say that on a set S, some property occurs in almost all cases if:

- for each case c at which this property is satisfied, this property is also satisfied for all sufficiently close cases, and
- for each case c and for each value $\varepsilon > 0$, there exits an ε-close case which has the desired property.

In Euclidean space, the above two properties mean that the set S of all the points having the desired property is open and its closure coincides with the

whole space. In other words, the complement to this set is closed and nowhere dense – e.g., a 2-D surface in a 3-D space.

What we will prove. We will prove that in the set of all possible Lagrangians, for almost all of them it is possible to solve NP-complete problems in polynomial time. This proof will consist of two steps:

- first, we will provide a specific example of a discrete space-time Lagrangain for which such a solution is possible;
- then, we will use this example to prove the desired general result.

This makes us hope that solving NP-complete problem in polynomial time is possible in the actual discrete space-time. In physics, if some property is proven to be true for almost all cases, physicists usually conclude that it is true in the real world as well. This is true in statistical physics, where, from the fact that in almost all situations, heat cannot move from the colder body to the warmer one, physicists conclude that such a transition is simply impossible – although, from the purely mathematical viewpoint, if we simply invert the directions of all the molecules' velocities in the normal warmer-to-colder heat transfer, we will see exactly this counterintuitive phenomenon. Similarly, in cosmology, from the proven result that almost all space-time models in General Relativity Theory have a singularity, physicists conclude that there is a singularity – e.g., the start of the Big Bang – in our Universe; see, e.g., [8, 24].

Following the same reasoning, from our result about almost all Lagrangians, we can conclude – with the same level of confidence with which physicists make their conclusions – that solving NP-complete problems in polynomial time is possible in the actual discrete space-time (provided, of course, that the actual space-time is indeed discrete).

6.2 An Example of a Discrete Space-Time Lagrangian L_0 That Enables Us to Solve NP-Complete Problems in Polynomial Time

Main idea. In this subsection, we will provide an example of a reasonable discrete-space-time Lagrangian – in the sense described in the previous section – for which the corresponding minimization is NP-complete. Thus, if we can implement this Lagrangian, then, by setting up proper initial conditions and observing the future state of the system, we will get a solution to an NP-complete problem.

By definition, as we have mentioned, a problem is NP-complete if every other problem from the class NP can be reduced to this problem. So, if

we have a polynomial-time algorithm for solving one NP-complete problem, then, by applying the corresponding reduction, we will be able to solve *all* the problems from the class NP in polynomial (i.e., feasible) time.

Let us now write down the desired Lagrangian.

The Lagrangian: the formula. Let us consider the case when we have three fields: $f(x,t)$, $\varepsilon(x,t)$, and $z(x,t)$. Let us consider the following Lagrangian:

$$L_0(x_i,t_j) = [f(x_i,t_j) - (f(x_{i-1},t_j) + \varepsilon(x_i,t_j)f(x_i,t_{j-1}))]^2 + \left[\varepsilon^2(x_i,t_j) - 1\right]^2 + [z(x_i,t_{j-1})f(x_i,t_j)]^2. \quad (2)$$

In other words, nature minimizes the following action:

$$S = \sum_{i,j} L_0(x_i,t_j). \quad (3)$$

Comment. This particular Lagrangian does not have any specific physical meaning, it is just an example for a lemma that we will use to prove the main result. Instead of L_0, we could use other Lagrangians for which, by using the corresponding physical fields, we can solve NP-complete problems in polynomial time. The only reason for selecting this particular Lagrangian L_0 is that for L_0, this fact is (relatively) easy to prove – it takes less than two pages.

How to use this Lagrangian to solve the corresponding NP-complete problem: idea. Let us assume that we are given an instance of the above problem (1), i.e., we are given n positive integers $s_1,\ldots,s_n$. We want to find the values $\varepsilon_1,\ldots,\varepsilon_n \in \{-1,1\}$ that satisfy the equation (1).

To find these values ε_i, at the current moment of time t_0, we set the following initial conditions:

- we set $f(x_i,t_0) = s_i$ for $i = 1,\ldots,n$ and $f(x_i,t_0) = 0$ for all other i;
- we also set $z(x_0,t_0) = z(x_n,t_0) = 1$ and $z(x_i,t_0) = 0$ for all other i.

As we have mentioned when we described the general minimal action principle, in terms of action S, the dynamics of a physical system works as follows:

- we have (or we set up) the initial conditions of all the corresponding fields,
- then, nature automatically sets up future values of all these fields in such a way as to minimize the value of the corresponding action S.

In line with this general description, once we set up the initial values of the corresponding three fields, nature will automatically set up the values of these fields at the next moment of time t_1 so as to minimize the expression (3).

Let us show that if the original instance of the NP-hand problem (1) has a solution, then this solution will appear as values $\varepsilon_i = \varepsilon(x_i, t_1)$. In other words, we will have

$$\sum_{i=1}^{n} \varepsilon(x_i, t_1) s_i = 0. \tag{4}$$

Indeed, if the current instance of the problem (1) has a solution $\varepsilon_1, \ldots, \varepsilon_n$, then we can take $\varepsilon(x_i, t_1) = \varepsilon_i$ and $f(x_i, t_1) = \sum\limits_{k=1}^{i} \varepsilon_k s_k$. In this case,

- for all i, we have

$$f(x_i, t_1) = \sum_{k=1}^{i} \varepsilon_k s_k =$$

$$\left(\sum_{k=1}^{i-1} \varepsilon_k s_k \right) + \varepsilon_i s_i = f(x_{i-1}, t_1) + \varepsilon(x_i, t_1) f(x_i, t_0);$$

  thus, the first term in the expression (2) is equal to 0;

- for all i, we have $\varepsilon(x_i, t_1) = \pm 1$, hence $\varepsilon^2(x_i, t_1) = 1$ and therefore, the second term in the expression (2) is equal to 0;

- for all $i \neq 0$ and $i \neq n$, we have $z(x_i, t_0) = 0$ hence

$$z(x_i, t_0) f(x_i, t_1) = 0;$$

- for $i = 0$, we have $f(x_0, t_1) = 0$ thus also $z(x_0, t_0) f(x_0, t_1) = 0$;

- for $i = n$, we have $f(x_n, t_1) = \sum\limits_{k=1}^{n} \varepsilon_k s_k = 0$ (since the values ε_i solve the problem (1)), thus also $z(x_n, t_0) f(x_n, t_1) = 0$; so, the third term in the Lagrangian is also equal to 0.

Thus, the Lagrangian is equal to 0, so the action is equal to 0 – and since the action is the sum of squares, 0 is its smallest possible value. So, if the given instance of the original NP-complete problem has a solution, the action can reach its smallest possible value 0.

Let us prove that, vice versa, if the action reaches 0, this means that the values $\varepsilon(x_i, t_1)$ form the solution to the given instance of the original problem (1). Indeed, if the sum of squares is 0, this means that all the squares are

equal to 0 and hence, each of the squared terms in the expression (2) is equal to 0.

In particular, from the fact that $z(x_0, t_0) f(x_0, t_1) = 0$, taking into account that $z(x_0, t_0) = 1$, we conclude that $f(x_0, t_1) = 0$. Now, for each i, from the equality

$$f(x_i, t_1) - (f(x_{i-1}, t_1) + \varepsilon(x_i, t_1) f(x_i, t_0)) = 0,$$

taking into account that $f(x_i, t_0) = s_i$, we conclude that

$$f(x_i, t_1) = f(x_{i-1}, t_1) + \varepsilon(x_i, t_1) s_i.$$

Since for $i = 0$, we have $f(x_0, t_1) = 0$, we can thus prove, by induction, that for all i, we have $f(x_i, t_1) = \sum_{k=1}^{i} \varepsilon(x_k, t_1) s_k$. In particular, for $i = n$, we get

$$f(x_n, t_1) = \sum_{k=1}^{n} \varepsilon(x_k, t_1) s_k.$$

From the fact that $z(x_n, t_0) f(x_n, t_1) = 0$, taking into account that $z(x_n, t_0) = 1$, we conclude that $f(x_n, t_1) = 0$, i.e., that

$$\sum_{k=1}^{n} \varepsilon(x_k, t_1) s_k = 0. \tag{5}$$

From the equality $\varepsilon^2(x_i, t_1) - 1 = 0$, we conclude that $\varepsilon^2(x_i, t_1) = 1$ hence $\varepsilon(x_i, t_1) \in \{-1, 1\}$ and thus, that the values $\varepsilon_i = \varepsilon(x_i, t_1)$ indeed solve the original instance of the above NP-complete problem.

The statement is proven.

What if the original problem (1) does not have a solution? In this case, we cannot have the equality (4) for $\varepsilon_i \in \{-1, 1\}$. Thus, we will either have $\varepsilon_i \notin \{-1, 1\}$ for some i, or the equality (4) will not be satisfied.

Now that we covered all possible cases, let us explicitly describe how we will be able to solve an NP-complete problem (1) in polynomial time – actually, in one step (if we do not count the time needed to set up the appropriate initial conditions).

How to use physical fields corresponding to the Lagrangian L_0 to actually solve every instance of the NP-complete problem (1) in polynomial time: algorithm. Suppose that we are given n positive integers $s_1, \ldots, s_n$, and we want to find the values $\varepsilon_i \in \{-1, 1\}$ for which $\sum_{i=1}^{n} \varepsilon_i s_i = 0$.

To find such values, at the initial moment of time t_0, we set up the following initial conditions:

- we set $f(x_i, t_0) = s_i$ for $i = 1, \ldots, n$ and $f(x_i, t_0) = 0$ for all other i;
- we also set $z(x_0, t_0) = z(x_n, t_0) = 1$ and $z(x_i, t_0) = 0$ for all other i.

At the next moment of time t_1, we measure the values $\varepsilon_i = \varepsilon(x_i, t_1)$ of the field $\varepsilon(x, t)$ at the points $x_1, \ldots, x_n$. Then we check whether the following two conditions are satisfied:

- that for each i, we have $\varepsilon_i \in \{-1, 1\}$ and
- that $\sum\limits_{i=1}^{n} \varepsilon_i s_i = 0$.

If these two conditions are satisfied, this means that we have the desired solution.

Vice versa, we have proven that if the corresponding instance of the problem (1) has a solution, then the values $\varepsilon(x_i, t_1)$ provide such a solution. Thus, if at least one the two above conditions is not satisfied, we can simply conclude that the corresponding instance of the problem (1) does not have a solution.

6.3 From the Example to a General Proof

What we want to prove: reminder. We want to prove that there exists a class $\mathcal{L}$ of Lagrangians with the following three properties:

- for each Lagrangian from this class, it is possible to solve an NP-complete problem in polynomial time;
- for each Lagrangian L from this class $\mathcal{L}$, all Lagrangians from some neighborhood of this Lagrangian also belong to $\mathcal{L}$; and
- for each Lagrangian L (not necessarily from the class $\mathcal{L}$) and for each $\varepsilon > 0$, there exists an ε-close Lagrangian L' from the class $\mathcal{L}$.

What this class $\mathcal{L}$ will include: first approximation. It is reasonable to require that the desired class $\mathcal{L}$ include the above Lagrangian L_0 – as well as all Lagrangians of the type $\varepsilon\, L_0$ with $\varepsilon > 0$, since for them, minimization is equivalent to minimizing L_0 and thus, enables us to solve NP-complete problems.

It also makes sense for this class $\mathcal{L}$ to include Lagrangians of the type $\varepsilon\, L_0 + L_1$, where L_0 depends only on the above three fields, while L_1 depends

only on some other fields – and does not depend on the above three fields at all. Indeed, since L_1 does not depend on the above three fields, minimizing the action

$$S = \sum_{i,j} (\varepsilon\, L_0(x_i, t_j) + L_1(x_i, t_j)) = \varepsilon \sum_{i,j} L_0(x_i, t_j) + \sum_{i,j} L_1(x_i, t_j)$$

with respect to these three fields is equivalent to minimizing the L_0-action $\sum\limits_{ij} L_0(x_i, t_j)$ and thus, as we have shown in the previous subsection, helps to solve NP-complete problems in polynomial time.

Already for this class – and thus, for each larger class – we have the property that for each Lagrangian L and for each $\varepsilon > 0$, there exists an ε-close Lagrangian L' from the class $\mathcal{L}$: indeed, we can simply add three new fields and take a Lagrangian $L' = L + c\varepsilon\, L_0$ for an appropriate constant c, where L_0 is described by the above formula in terms of the three new fields.

Towards a full description of the class $\mathcal{L}$. In the previous subsection, we showed that the Lagrangian L_0 helps solve the following NP-complete problem:

- given positive integers $s_1, \ldots, s_n$,
- find the values $\varepsilon_1 \in \{-1, 1\}, \ldots, \varepsilon_n \in \{-1, 1\}$ for which the following equality holds:

$$\sum_{i=1}^{n} \varepsilon_i\, s_i = 0.$$

It is known that this problem is NP-complete, and moreover, it is known that this NP-completeness is *stable* in the following sense: if instead of the equality, we require a weaker condition – of inequality

$$-\delta \le \sum_{i=1}^{n} \varepsilon_i\, s_i \le \delta,$$

then the problem remains NP-complete for all sufficiently small δ – namely, for all $\delta < 1$. Indeed, since all the values s_i and ε_i are integers, the sum $\sum\limits_{i=1}^{n} \varepsilon_i\, s_i$ is also an integer – and for $\delta < 1$, the only integer between $-\delta$ and δ is 0.

Moreover, the problem remains NP-complete if in addition to relaxing the condition $\sum\limits_{i=1}^{n} \varepsilon_i\, s_i = 0$, we also:

- instead of looking for values ε_i whose absolute values are equal to 1,
- look for values for which $1-\alpha < |\varepsilon_i| < 1+\alpha$.

Then, for sufficiently small $\alpha > 0$ and $\delta > 0$, the problem remains NP-complete. Indeed, because of the requirement on ε_i, for each i, there exists a value $\varepsilon_i' \in \{-1, 1\}$ which is α-close to ε_i. Here,

$$\sum_{i=1}^{n} \varepsilon_i\, s_i - \sum_{i=1}^{n} \varepsilon_i'\, s_i = \sum_{i=1}^{n} (\varepsilon_i - \varepsilon_i')\, s_i.$$

Thus,

$$\left| \sum_{i=1}^{n} \varepsilon_i\, s_i - \sum_{i=1}^{n} \varepsilon_i'\, s_i \right| = \left| \sum_{i=1}^{n} (\varepsilon_i - \varepsilon_i')\, s_i \right| \le \sum_{i=1}^{n} |(\varepsilon_i - \varepsilon_i')\, s_i|.$$

Here, $|\varepsilon_i - \varepsilon_i'| \le \alpha$, hence

$$\left| \sum_{i=1}^{n} \varepsilon_i\, s_i - \sum_{i=1}^{n} \varepsilon_i'\, s_i \right| \le \alpha \sum_{i=1}^{n} s_i.$$

Since

$$\left| \sum_{i=1}^{n} \varepsilon_i\, s_i \right| \le \delta,$$

we can thus conclude that

$$\left| \sum_{i=1}^{n} \varepsilon_i'\, s_i \right| \le \left| \sum_{i=1}^{n} \varepsilon_i\, s_i - \sum_{i=1}^{n} \varepsilon_i'\, s_i \right| + \left| \sum_{i=1}^{n} \varepsilon_i\, s_i \right| \le \alpha \sum_{i=1}^{n} s_i + \delta.$$

For sufficiently small α and δ, the sum $\alpha \sum\limits_{i=1}^{n} s_i + \delta$ is smaller than 1 and thus, the only way to satisfy the above inequality with integer values ε_i' is to have $\sum\limits_{i=1}^{n} \varepsilon_i'\, s_i = 0$ – which is an NP-complete problem.

Because of this stability, we can conclude that for each Lagrangian of the type $L' \stackrel{\text{def}}{=} \varepsilon\, L_0 + L_1$, there exist a small $\beta > 0$ for which, for all Lagrangians L whose distance to L' is smaller than β, we can also solve the above NP-complete problem in polynomial time. Now, as $\mathcal{L}$, we will take the class of all such "close" Lagrangians L.

The first of the desired three properties is satisfied by definition of the class. The third property is satisfied because it was already satisfied for Lagrangians of the type $\varepsilon\, L_0 + L_1$, and such Lagrangians – as one can easily check – belong to the above-defined class $\mathcal{L}$.

The second property is also easy to check. Indeed, for each Lagrangian L from the above-defined class, there exists a Langangian L' of the type $\varepsilon\, L_0 + L_1$ for which $d(L, L') < \beta$, i.e., for which $\beta - d(L, L') > 0$. Thus, for each Lagrangian L'' for which $d(L'', L) \le \beta - d(L, L')$, we have

$$d(L'', L') \le d(L'', L') + d(L'L) < (\beta - d(L, L')) + d(L, L') = \beta,$$

i.e., $d(L'', L') < \beta$. Thus, by definition of the class $\mathcal{L}$, we have $L'' \in \mathcal{L}$. So, all the Lagrangians from some neighborhood of L indeed belong to the class $\mathcal{L}$ – which is exactly the second of the three desired properties.

So, our main result has been proven.

7 CONCLUSION

Main conclusion. Usually, physics considers continuous models of space-time. However, the very continuous nature of space-time leads to many still unresolved physical challenges and paradoxes. Several physicists suggested that a natural way to resolve these challenges is to consider discrete space-time models. In this paper, we show that this suggestion has an interesting consequence in terms of computability. Namely, we show that in almost all discrete space-time models, we can use physical phenomena to solve NP-complete problems in polynomial time.

In physics in general, when a property is true in almost all cases, it is usually concluded that this property is true in the real world as well. By using this logic, we can make an even stronger conclusion – that if the space-time is indeed discrete, then in the real world, we will be able to solve NP-complete problems in polynomial time.

What is the intuitive meaning of this result? Discretization of space-time can be viewed as particular example of quantization – indeed, quantum mechanics started by physicists realizing that electromagnetic waves of a given frequency (e.g., visible light) cannot have an arbitrary small energy: there is the smallest amount of such energy (quantum), and any other amount is proportional to this one. Quantum physics continued with observing that an atom cannot be in a state with any possible energy: there are discrete energies, etc.

From this viewpoint, our result about computing in a discrete space-time can be viewed as a particular case of – broadly understood – quantum computing; see, e.g., [18]. It is known that quantum effects can speed up computations. Our result shows that if we extend quantization to space-time, this speed-up can be further enhanced – to the point that it allows us to solve NP-complete problems in polynomial time.

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